3D VISION

Projective Model of a Stereo Binocular Camera
Fundamental Matrix

Systems Engineering and Automation

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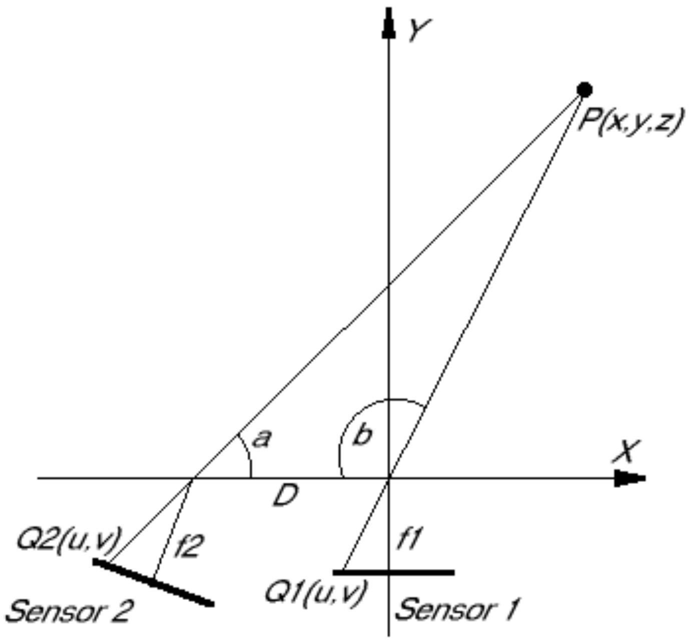
Introduction

- 3D Vision attempts to estimate the spatial information of the world through a set of images taken from different points of view
  - *Binocular* Disposition (Passive)
  - *Trinocular* Disposition(Passive)
  - *Sequence of Images* (Active)

Introduction

- Stereoscopic Pair (Binocular)
  - The third dimension is retrieved by triangulation from the scene observed by two or more catchers (Disparity)

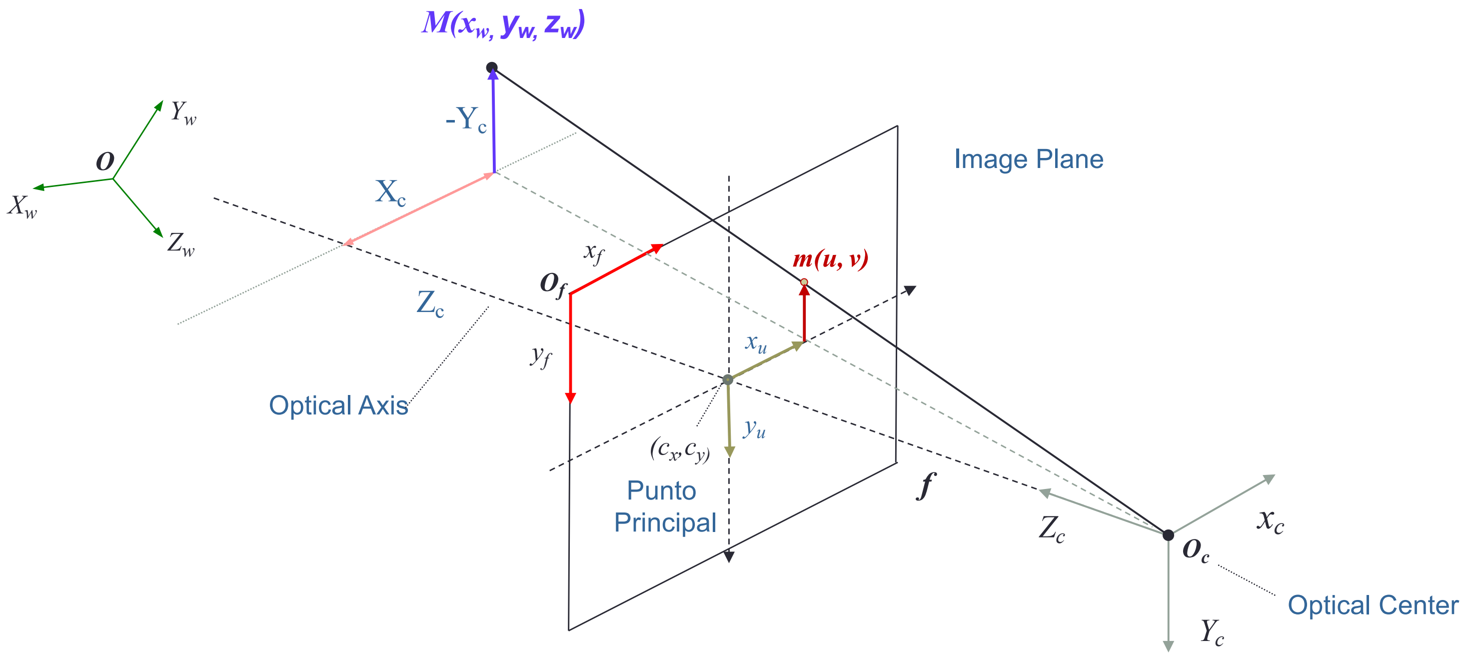


Introduction

- Fundamental problems in stereo vision:
  - Calibration:
    - Obtaining the projection model of both cameras
  - Preprocessing:
    - Determining characteristic features in each image.
  - Correspondence (matching):
    - Which element in the left camera corresponds to which element in the right camera?
  - 3D reconstruction:
    - Difference in the position of the corresponding points (Disparity).
    - If the geometry of the stereo system is known, then we can build a 3D map of the scene (Triangulation)

Introduction

- Projective Model of a Camera:



Introduction

- Projective Model of a Camera :

$$\tilde{m}_f = \tilde{A}\ \tilde{M}_c^w\ \tilde{M}_w \qquad\qquad \tilde{M}_c = \tilde{M}_c^w\ \tilde{M}_w$$

$$\tilde{A} = \underset{3x3}{[A} \quad \underset{3x1}{0]} \qquad\qquad \tilde{M}_c{}^w = \begin{bmatrix} R_c{}^w & t_c{}^w \\ 0 & 1 \end{bmatrix}$$

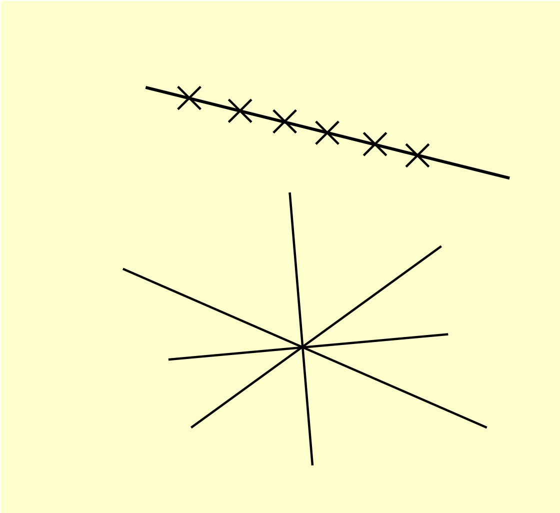
$$\boxed{\tilde{m} \cong A\ [R_c^w \quad t_c^w]\ \tilde{M}}$$

$$\boxed{\tilde{m} \cong A R_c^w\ [I \quad -C]\ \tilde{M}} \qquad \boxed{C = -R_c^{w\,T} t_c^w}$$

Projective Plane P^2

- Point: $\tilde{x} \in P^2 \Rightarrow \tilde{x} \cong [x_1, x_2, x_3]^T$
- Line: $\tilde{l} \in P^2 \Rightarrow \tilde{l} \cong [l_1, l_2, l_3]^T$
- Line passing through a point : $\tilde{l}^{\,T} \cdot \tilde{x} = 0$

- Duality between points and lines:
  - Set of points that are in a line
  - Set of lines passing through a point

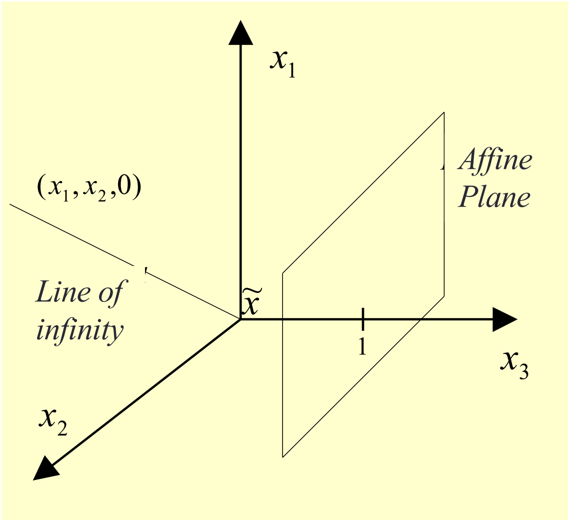


Projective Plane P^2

- Projective Plane P^2

$$if\ \ x_3 \neq 0; \quad \tilde{x} \cong [x_1, x_2, x_3]^T \Rightarrow \tilde{x} \cong \left[\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right]^T \cong [X_1, X_2, 1]^T$$

- This set of points defines the Affine Plane



The points with $x_3=0$ have yet to be considered. They define the line of infinity

Projective Plane P^2

- Projective Plane P^2
  - Belonging from a point to a line on the projective plane:

$$\tilde{l}^{\,T} \cdot \tilde{x} = 0 \quad \Rightarrow \quad l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$

  - Belonging from a point to a line in the affine plane:

$$l_1 X_1 + l_2 X_2 + l_3 = 0$$

  - Point to line distance:

$$dist(\tilde{x}, \tilde{l}) = \frac{\tilde{l}^{\,T} \cdot \tilde{x}}{\sqrt{l_1^2 + l_2^2}} \qquad \vec{n} \cong [l_1 \quad l_2]$$

Normal to the line

  - Belonging to the line that passes through two points: $\tilde{x}_1, \tilde{x}_2$

$$\tilde{x} \cong \alpha \tilde{x}_1 + \beta \tilde{x}_2 \quad or \quad \tilde{x} \cong \gamma \tilde{x}_1 + \tilde{x}_2$$

$$\text{line} \rightarrow \quad \tilde{l} \cong \tilde{x}_1 \wedge \tilde{x}_2 \quad \Rightarrow \quad \tilde{l}^{\,T} \cdot \tilde{x} = 0$$

Projective Plane *P*²

- Vector Cross Product:

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \qquad [\tilde{x}]_\wedge = \begin{bmatrix} 0 & -\tilde{x}_3 & \tilde{x}_2 \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ -\tilde{x}_2 & \tilde{x}_1 & 0 \end{bmatrix}$$

$$\tilde{l} \cong \tilde{x} \wedge \tilde{y} = [\tilde{x}]_\wedge \tilde{y}$$

Projective Plane *P*²

- Projective Plane P^2
  - Collineation in the projective plane (Form a projective group):

$$\rho\,\tilde{y} = \tilde{A}\,\tilde{x} \quad \Rightarrow \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

  - Affine Subgroup (Holds the line at infinity): $\tilde{A} = \begin{pmatrix} a_{33}B_{2x2} & a_{33}b_{2x1} \\ 0_{1x2} & a_{33} \end{pmatrix}$

  - Similarity Transformation: $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = c \begin{pmatrix} \cos\theta & sen\theta \\ -sen\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + b_{2x1}$

    rotates (θ) Scales (c) Traslates (b)

  - Euclidean Transformation: Similarity Transformation with $c=1$

Projective Plane P^2

- Homographic Correspondence or Homography:
  - One-to-one correspondence (linear and invertible). It can be between:
    - Point-to-Point
    - Line-to-Line
    - Line Pencil to Line Pencil

$$\tilde{m}' \cong \tilde{H}\tilde{m}$$

- Correlation:
  - Transform lines into points and vice versa

$$\tilde{l} \cong \tilde{F}\tilde{x}$$

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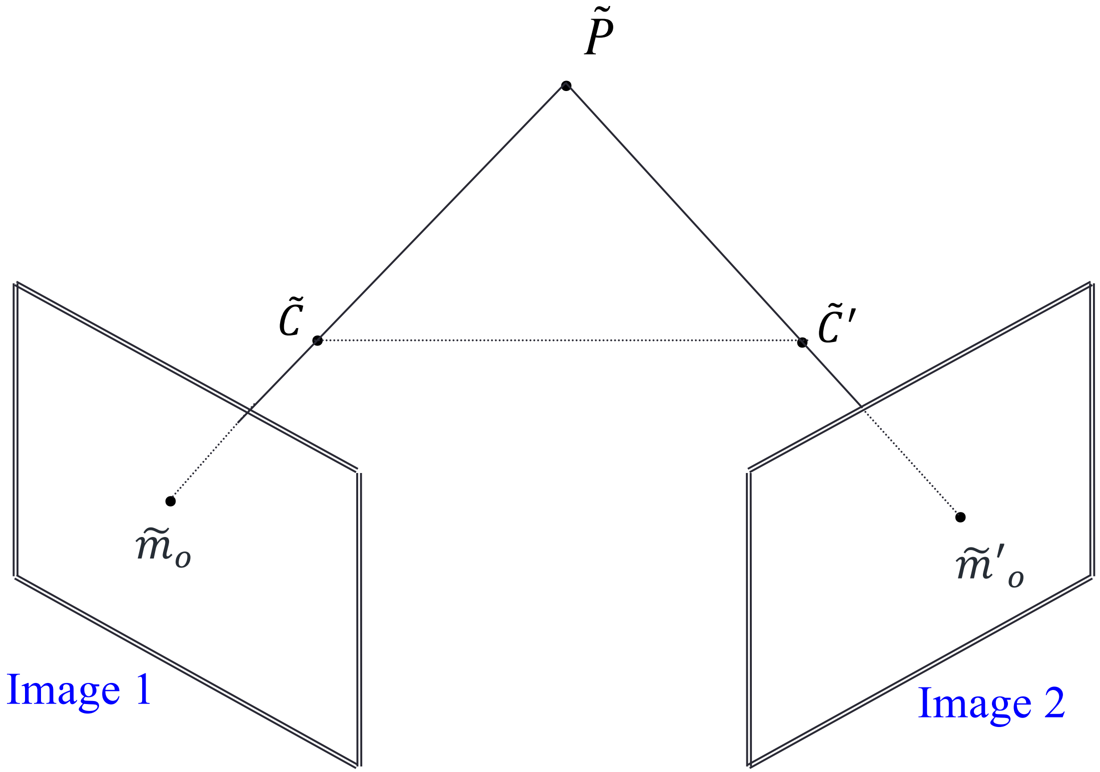
Binocular Vision

- The disposition of the optical axes can be:
  - *Convergent Axes*
  - *Parallel Axes*

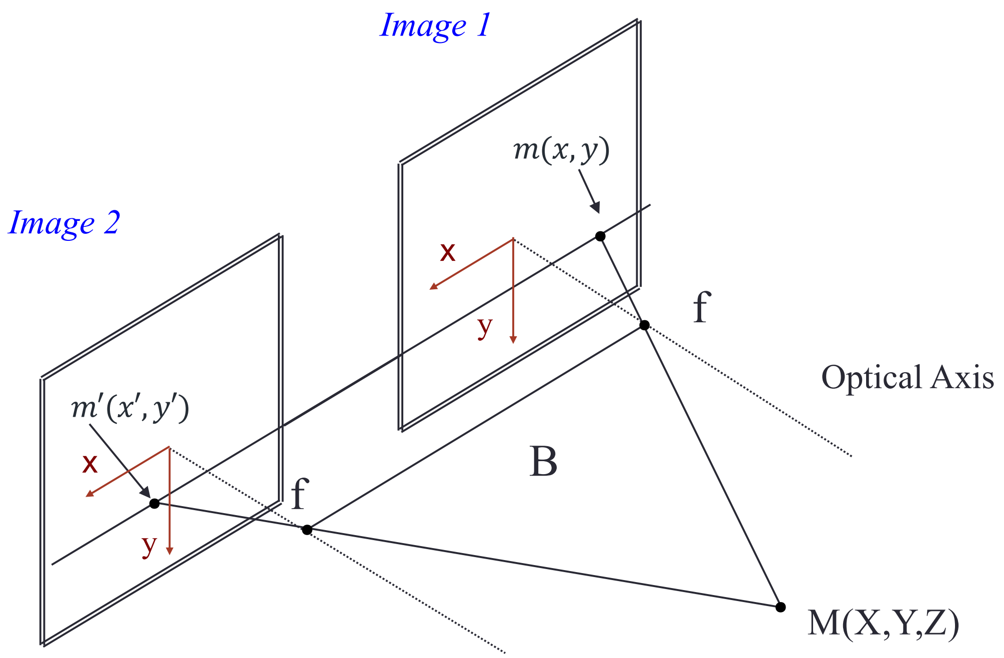
Binocular Vision

- Convergent Axes
  - Facilitates the correspondence process
  - Less occlusions



Binocular Vision

- Aligned Axles
  - Simplicity in the projective and reconstruction model
  - More problems with occlusions at close distances
  - Increased correspondence difficulty (greater disparity)



Projective Model of a Binocular Stereo Camera

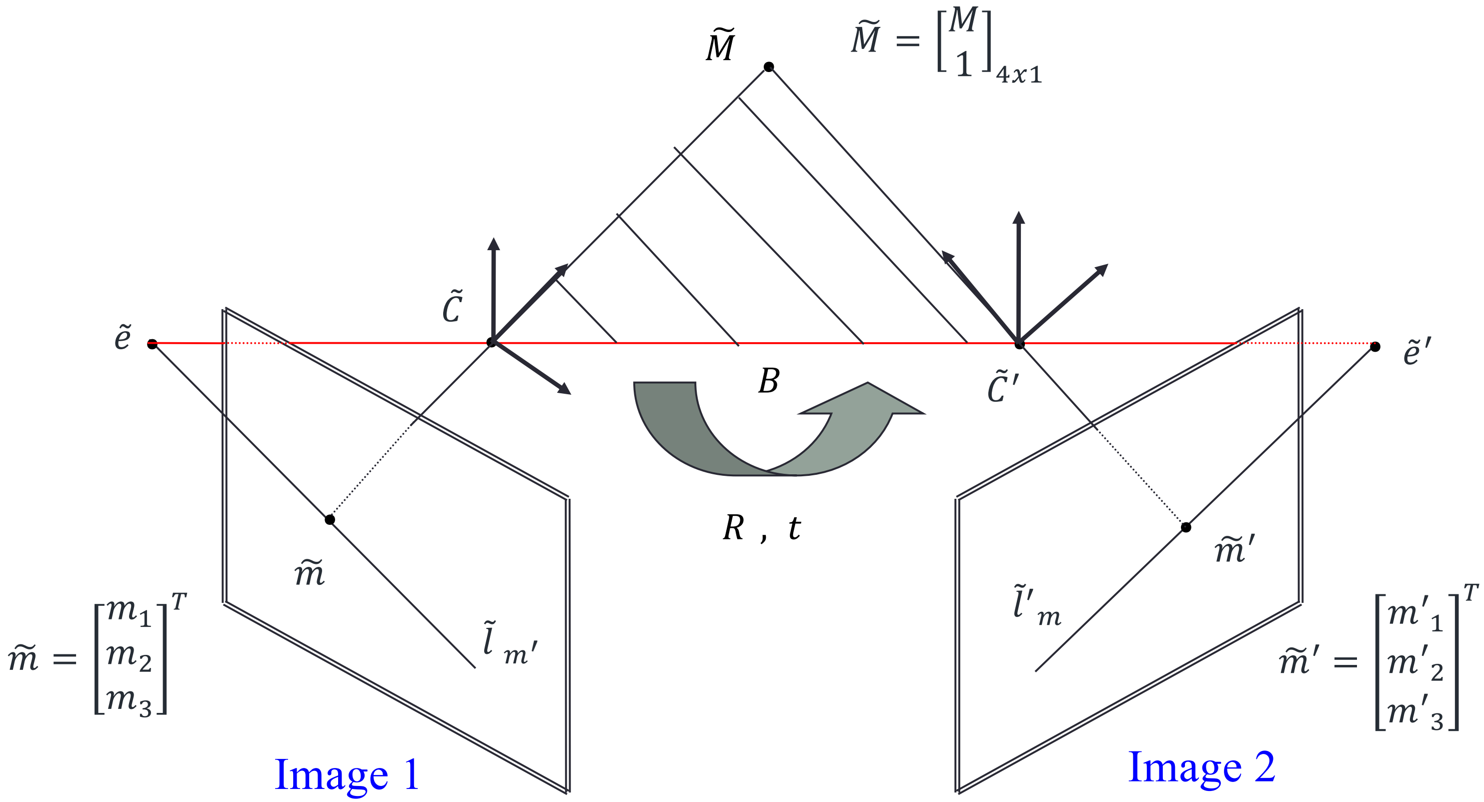
- Representation can be done:
  - In World Coordinates: $\tilde{m} \cong \tilde{A}\,\tilde{M}_c^w\,\tilde{M}_w$
    - It has an excessive dependence on the world's coordinate system, with terms that will be canceled
    - The geometric characteristics will depend on the parameters of each camera and the arrangement between them, but not on points of the world and its representation)
  - In Camera Coordinates:

$$\tilde{m} \cong \tilde{A}\,\tilde{M}_c^w.\,\tilde{M}_w \cong \tilde{A}\,\tilde{M}_c$$

$$\text{con}\quad \tilde{M}_c^w = I \qquad \text{se define}\quad \tilde{M}_{c\prime}^c = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Projective Model of a Binocular Stereo Camera

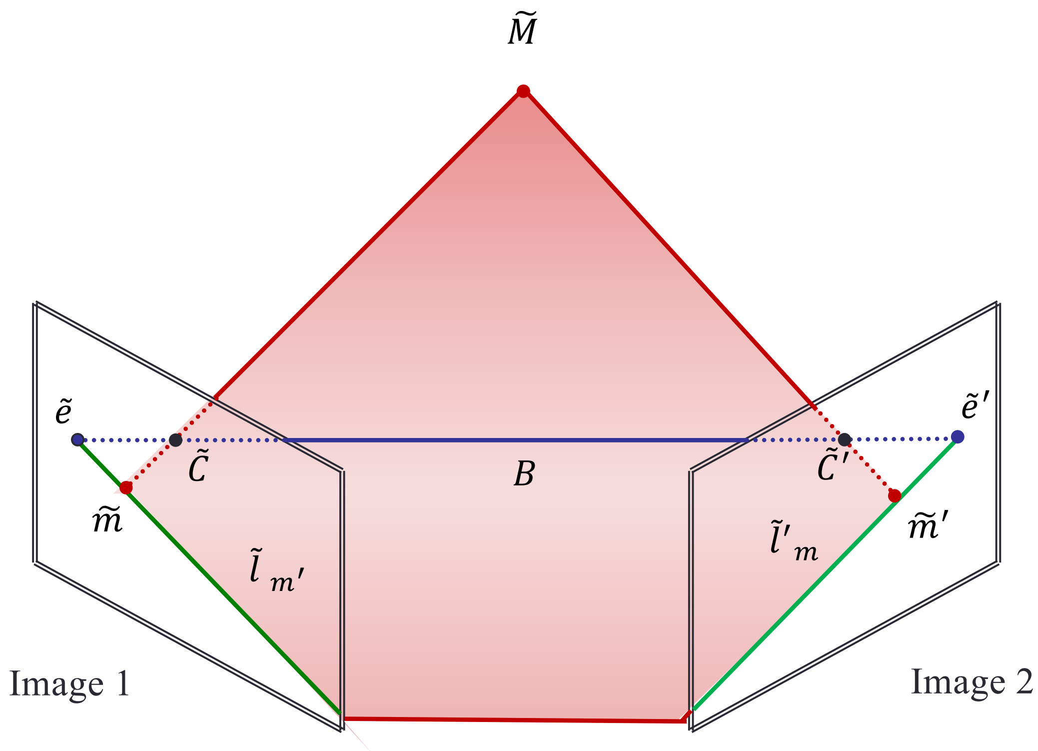
- Binocular disposition of convergent axes



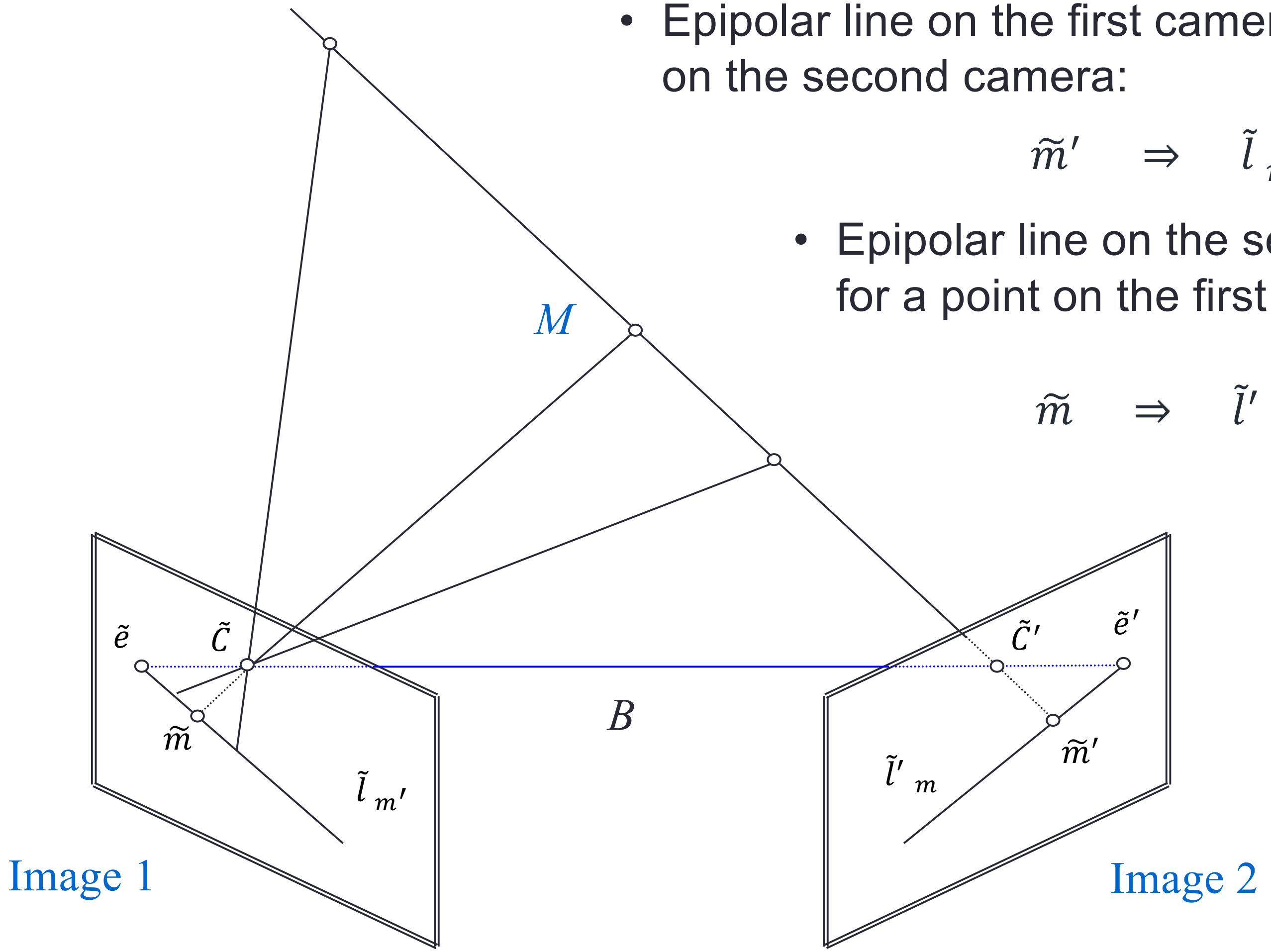
- Baseline: $[\tilde{C} \ , \ \tilde{C}']$
- Epipolar Plane: $[\tilde{M} \ , \ \tilde{C} \ , \ \tilde{C}']$

Projective Model of a Binocular Stereo Camera

- Epipolar Plane
- Epipolar lines
- Epipoles



Projective Model of a Binocular Stereo Camera



- Epipolar line on the first camera for a point on the second camera:

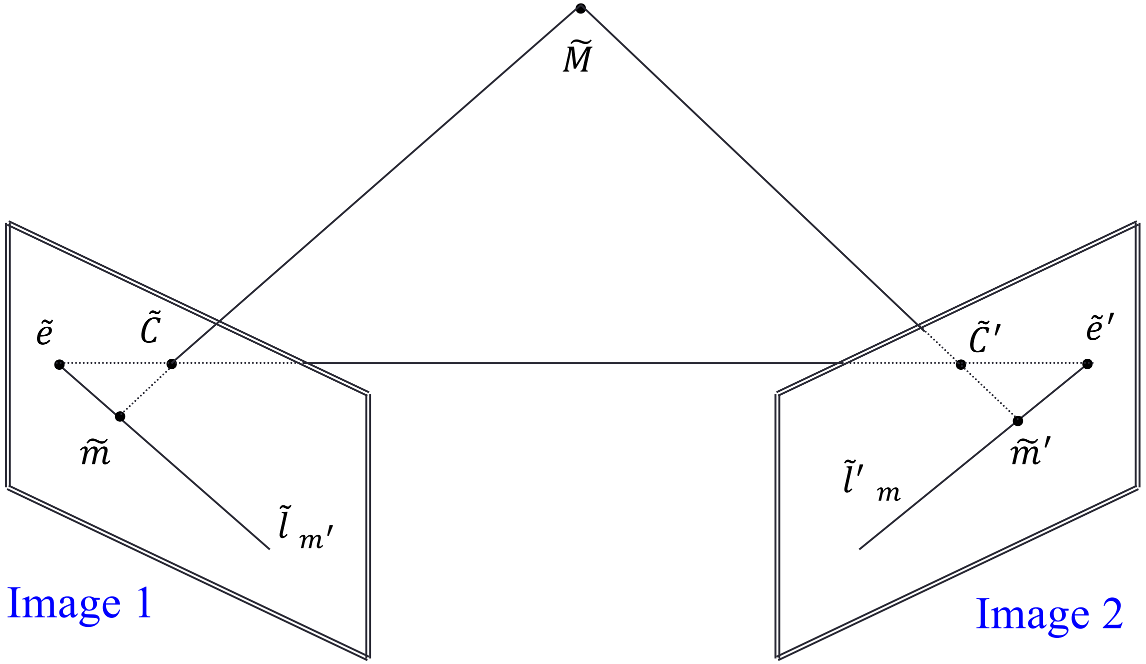
$$\tilde{m}' \quad \Rightarrow \quad \tilde{l}_{m'}$$

- Epipolar line on the second camera for a point on the first camera:

$$\tilde{m} \quad \Rightarrow \quad \tilde{l}'_{m}$$

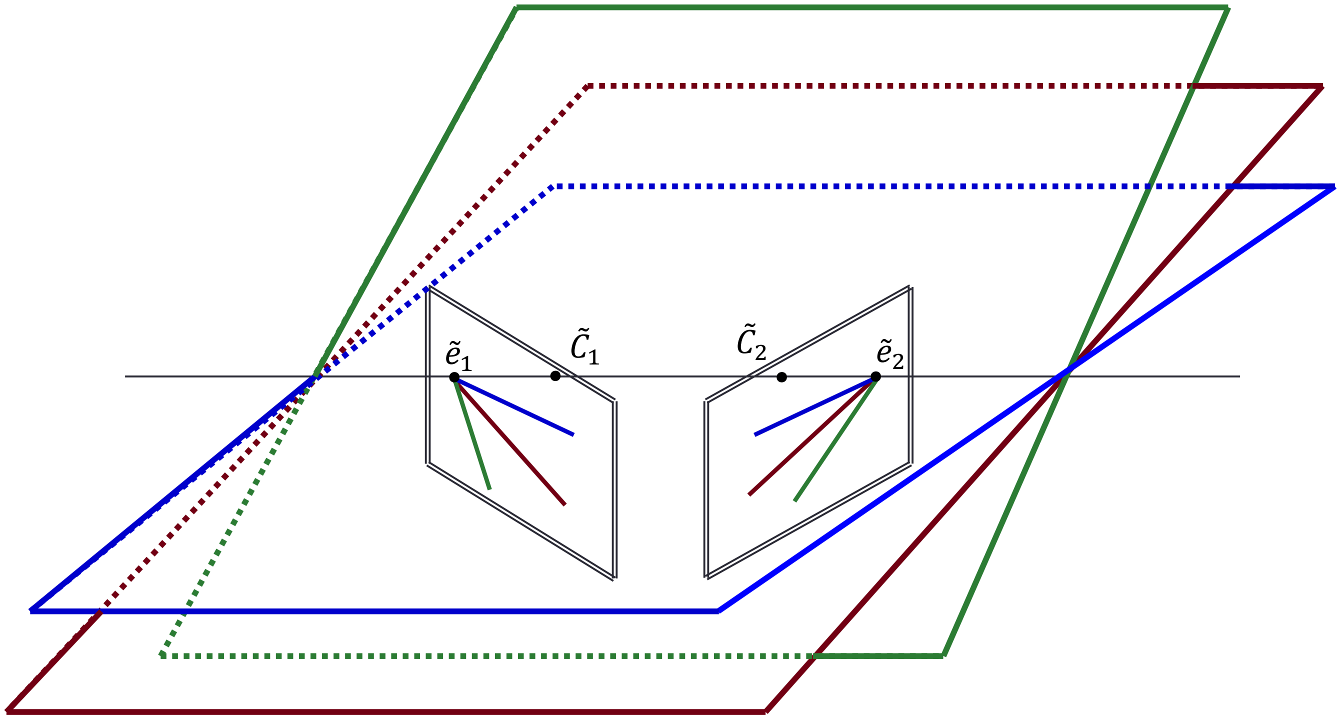
Projective Model of a Binocular Stereo Camera

- Epipoles:
  - Intersection point of all epipolar lines $\forall\ \widetilde{M}$
  - Projection of $\tilde{C}'$ on the first camera $\tilde{l}_{m'} \Rightarrow \tilde{e}$
  - Projection of $\tilde{C}$ on he second camera $\tilde{l}'_{m} \Rightarrow \tilde{e}'$



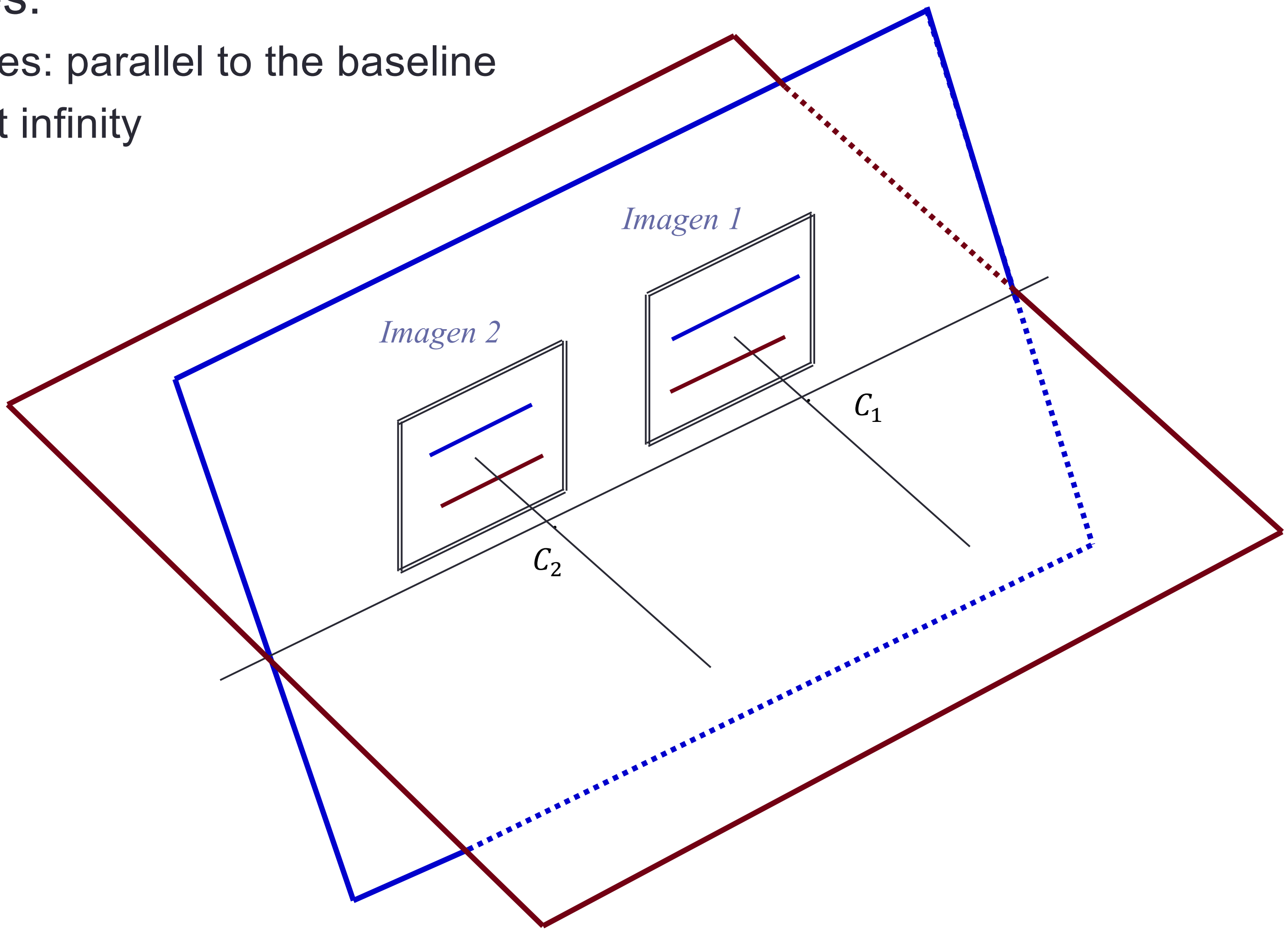
Pencil of Epipolar Planes

- Convergent Axes
  - Epipolar lines: intersection of epipolar planes over the image planes



Pencil of Epipolar Planes

- Aligned Axes:
  - Epipolar lines: parallel to the baseline
  - Epipoles: at infinity



Stereo Projective Model (Camera Coordinates)

- Equations of the Stereo Projective Model:
  - If the coordinates are referenced to the optical center of each camera, then:

$$\tilde{m} = \tilde{m}_f \cong \tilde{A}\ \tilde{M}_c \quad ; \quad \tilde{m}' = \tilde{m}'_f \cong \tilde{A}'\ \tilde{M}_{c'}$$

$$\tilde{A} = A[I \quad 0] \ ; \quad \tilde{A}' = A'[I \quad 0]$$

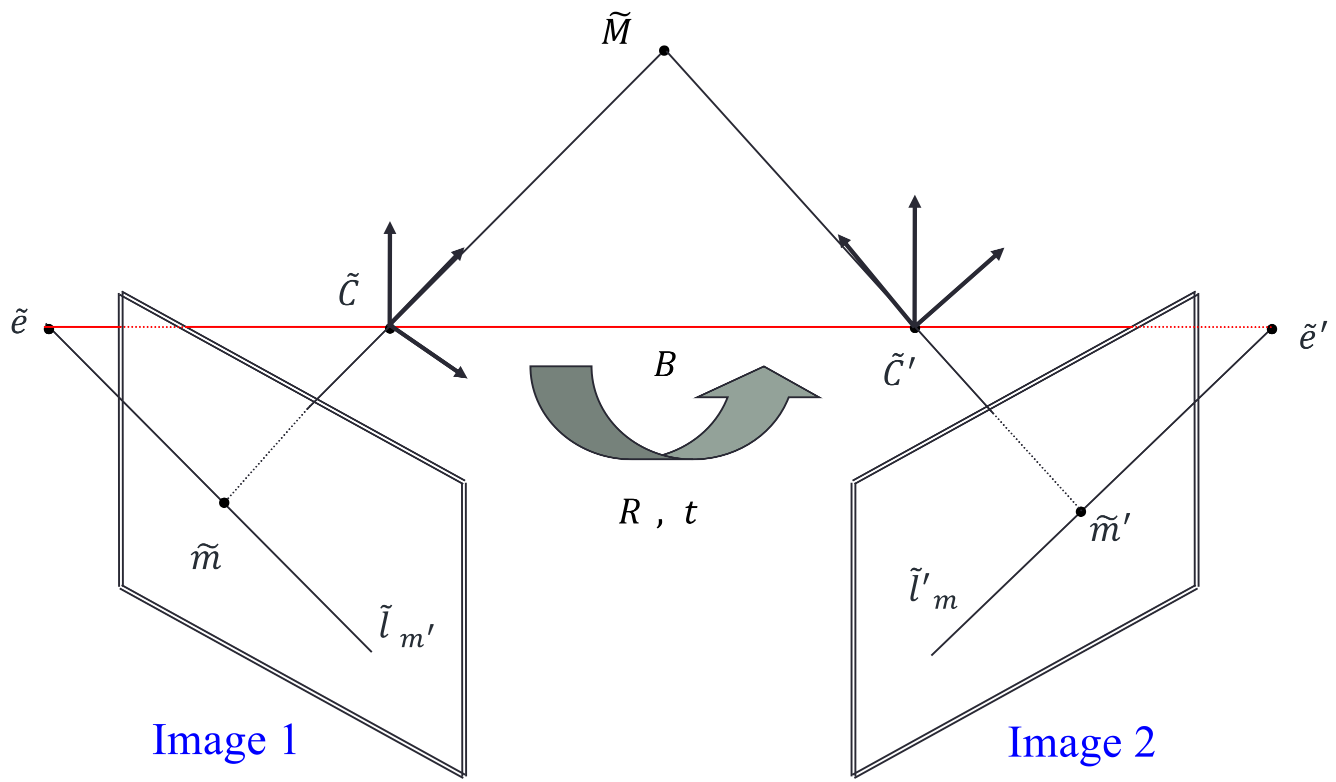
  - The two camera coordinate systems will be related by a rotation and a translation:

$$\tilde{M}_{c'} = \tilde{M}^{c}_{c'}\ \tilde{M}_c \qquad \text{con} \qquad \tilde{M}^{c}_{c'} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} ; \qquad \tilde{M}_c{}^{c'} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

  - For a 3D coordinate system centered on the left camera:

$$\boxed{\tilde{m} \cong A[I \quad 0]\ \tilde{M}_c \quad ; \quad \tilde{m}' \cong A'[R \quad t]\ \tilde{M}_c}$$

Stereo Projective Model (Camera Coordinates)



Stereo Projective Model (Camera Coordinates)

- Epipole of the first camera:
  - The projection on the first camera is:

$$\widetilde{m} \cong \tilde{A}\ \widetilde{M}_c = \tilde{A}\ \widetilde{M}_c^{c'}\ \widetilde{M}_{c'}$$

  - To find the epipole, we project of the optical center of the second camera on the first camera:

$$if \quad \widetilde{M}_{c'} \cong [0 \quad 0 \quad 0 \quad 1]^T \ \Rightarrow \ \widetilde{m} \cong \tilde{e}$$

$$\tilde{e} \cong \tilde{A} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = [A \quad 0] \begin{bmatrix} -R^T t \\ 1 \end{bmatrix} = -A\ R^T\ t$$

$$\boxed{\tilde{e} \ \cong \ A\ R^T\ t}$$

Stereo Projective Model (Camera Coordinates)

- Epipole of the second camera:
  - The projection on the second camera is:

$$\tilde{m}' \cong \tilde{A}' \, \widetilde{M}_{c'} = \tilde{A}' \, \widetilde{M}_{c'}^{c} \, \widetilde{M}_{c}$$

  - To find the epipole, we project of the optical center of the first camera on the second camera :

$$if \quad \widetilde{M}_{c'} \cong \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T \;\Rightarrow\; \tilde{m}' \cong \tilde{e}'$$

$$\tilde{e}' \cong \tilde{A}' \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} A' & 0 \end{bmatrix} \begin{bmatrix} t \\ 1 \end{bmatrix} = A' \, t$$

$$\boxed{\tilde{e}' \cong A' \quad t}$$

Stereo Projective Model (Camera Coordinates)

- Epipolar line in the second chamber for a point $\tilde{m}$

$$\boxed{\tilde{l}'_m \cong \tilde{e}' \wedge \tilde{m}'}$$

$$\tilde{m} \cong \tilde{A}\ \tilde{M}_c = [A \quad 0]\begin{bmatrix} M \\ 1 \end{bmatrix}$$

With: $\tilde{e}' \cong A'\ t$

$$\tilde{m} \cong A\ M$$

$$\tilde{m}' \cong [A' \quad 0]\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} M \\ 1 \end{bmatrix} = A'RM + A'\ t = A'R[\lambda A^{-1}\tilde{m}] + A'\ t$$

$$\tilde{l}'_m \cong \tilde{e}' \wedge \tilde{m}' = [A't] \wedge A'RA^{-1}\tilde{m} = A'^{-T}[t]_\wedge RA^{-1}\tilde{m}$$

$$\tilde{l}_{m'} \cong \tilde{e} \wedge \tilde{m} = [AR^T t] \wedge AR^T A'^{-1}\tilde{m}' = A^{-T}R^T[t]_\wedge A'^{-1}\tilde{m}'$$

$$\boxed{\tilde{l}_{m'} \cong A^{-T}R^T[t]_\wedge A'^{-1}\tilde{m}'} \qquad \boxed{\tilde{l}'_m \cong A'^{-T}[t]_\wedge RA^{-1}\tilde{m}}$$

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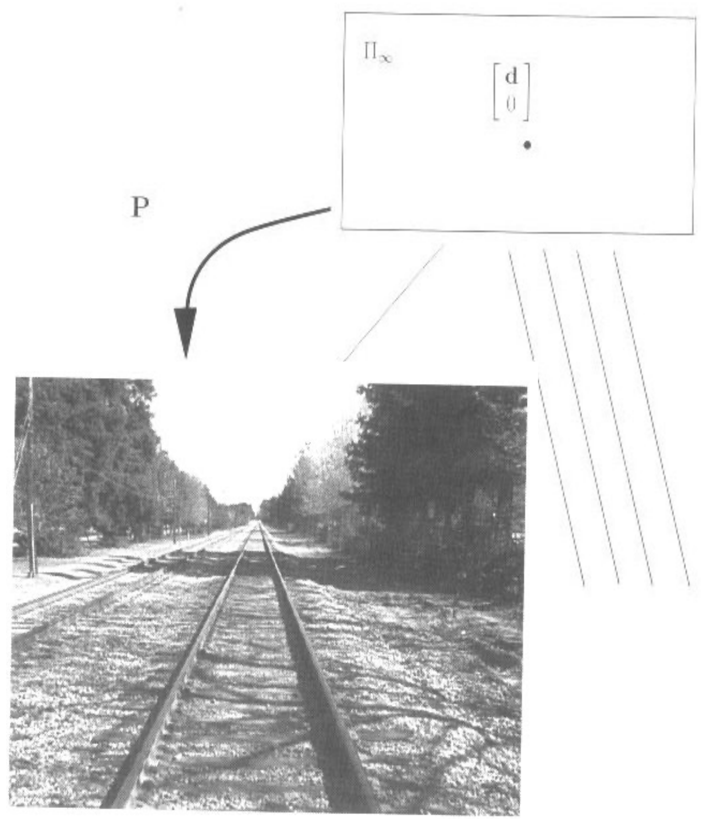
Homographies between Projective Planes

- Plane at infinity – Image

Projected point $\tilde{m} \cong \tilde{P}\,\tilde{M} = A[R \quad t]\tilde{M} = [P \quad p]\tilde{M}$

if $\tilde{M} \cong \begin{bmatrix} M \\ 0 \end{bmatrix} \in \prod_\infty \Rightarrow \tilde{m} \cong P\,M \Rightarrow M \cong P^{-1}\tilde{m}$

$$\boxed{M \cong H_\infty\,\tilde{m}, \quad H_\infty \cong P^{-1} = R^T \cdot A^{-1}}$$



- Projects intersections of parallel lines at vanishing points
- Infinity points (vanishing points) represent directions

Homography between points in the Infinity Plane

- Stereo Pair. Camera Coordinates:

$$\tilde{m} \cong \tilde{A}\ \tilde{M}_c = [A \quad 0]\ \tilde{M}_c \quad ; \quad \tilde{m}' \cong \tilde{A}'\ \tilde{M}_{c'} = [A' \quad 0]\ \tilde{M}_{c'}$$

$$\tilde{M}_{c'} = \tilde{M}_{c'}^{c}\ \tilde{M}_c \quad \text{con} \quad \tilde{M}_{c'}^{c} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \ ; \quad \tilde{M}_c{}^{c'} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

- **Projecting in Camera 1:** $\tilde{M}_c \cong \begin{bmatrix} M_c \\ 0 \end{bmatrix} \in \prod_\infty \Rightarrow \tilde{m} \cong [A \quad 0] \begin{bmatrix} M_c \\ 0 \end{bmatrix} = A\ M_c \Rightarrow \quad M_c \cong A^{-1}\tilde{m}$

- **Projecting in Camera 2:** $\tilde{M}_c \cong \begin{bmatrix} M_c \\ 0 \end{bmatrix} \in \prod_\infty$ in the second camera $\tilde{m}' \cong \tilde{A}'\ \tilde{M}_{c'}$

$$\tilde{m}' \cong \tilde{A}'\ \tilde{M}_{c'}^{c} \begin{bmatrix} M_c \\ 0 \end{bmatrix} = [A' \quad 0] \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} M_c \\ 0 \end{bmatrix} \Rightarrow \quad \tilde{m}' \cong A' R M_c$$

$$\tilde{m}' \cong A' R A^{-1} \tilde{m} \quad \Rightarrow \quad \tilde{m}' \cong \tilde{H}_\infty \tilde{m} \quad \text{with} \quad \boxed{\tilde{H}_\infty \cong A' R A^{-1}}$$

Homography between Points of a Plane

- Image Plane - Image Plane
  - Plane Π doesn't go through the optical center ***C***

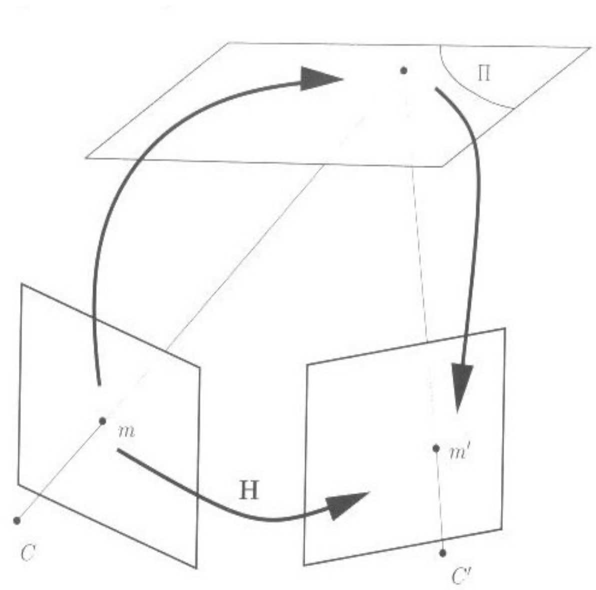
$$\tilde{m} \cong \tilde{P}\ \tilde{M} \quad ; \quad \tilde{m}' \cong \tilde{P}'\ \tilde{M}$$

$$\text{Si}\ \ \tilde{M} \cong \begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix} \in \prod \Rightarrow \tilde{m} \cong \begin{bmatrix} P_{11} & P_{12} & P_{14} \\ P_{21} & P_{22} & P_{24} \\ P_{31} & P_{32} & P_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

$$\tilde{m} \cong \tilde{H}_1\ \tilde{M}_{\Pi}, \qquad \tilde{M}_{\Pi} \cong \tilde{H}_1^{\ -1}\tilde{m}$$

$$\tilde{m}' \cong \tilde{H}_2\ \tilde{M}_{\Pi} = \tilde{H}_2\ \tilde{H}_1^{\ -1}\tilde{m}$$

$$\boxed{\tilde{m}' \cong \tilde{H}_{\Pi}\ \tilde{m}}$$



Homography between Points of a Plane

- $\tilde{H}_{\pi}$ has 8 degrees of freedom (nine elements minus one for scaling)
- It can be obtained by from the projections of four points that are a projective basis:

Let be $\tilde{M}_i \in \Pi$ with $i = 1, \cdots, 4$

If we identify $\tilde{m}_i, \tilde{m}'_i$ with $\tilde{m}_i = \tilde{P}\tilde{M}_i$; $\tilde{m}'_i = \tilde{P}'\tilde{M}_i$

We can get $\tilde{H}_{\Pi}$

$$\begin{bmatrix} \rho \cdot x' \\ \rho \cdot y' \\ \rho \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{A}_{xy} = \begin{bmatrix} -x^{(1)} & -y^{(1)} & -1 & 0 & 0 & 0 & x^{(1)}x'^{(1)} & y^{(1)}x'^{(1)} & x'^{(1)} \\ 0 & 0 & 0 & -x^{(1)} & -y^{(1)} & -1 & x^{(1)}y'^{(1)} & y^{(1)}y'^{(1)} & y'^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{h} = [h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33}]^T$$

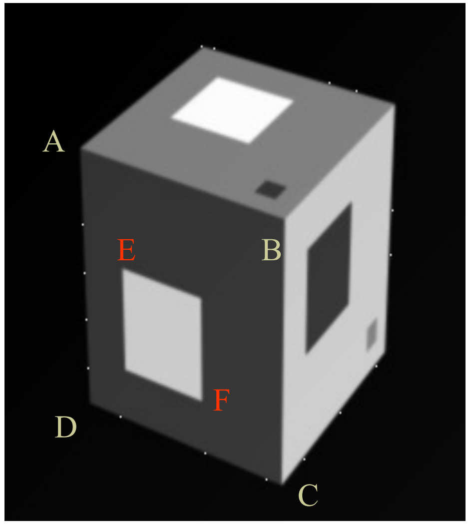
$$(h_{31}x + h_{32}y + h_{33}) \cdot x' = h_{11}x + h_{12}y + h_{13}$$
$$(h_{31}x + h_{32}y + h_{33}) \cdot y' = h_{21}x + h_{22}y + h_{23}$$

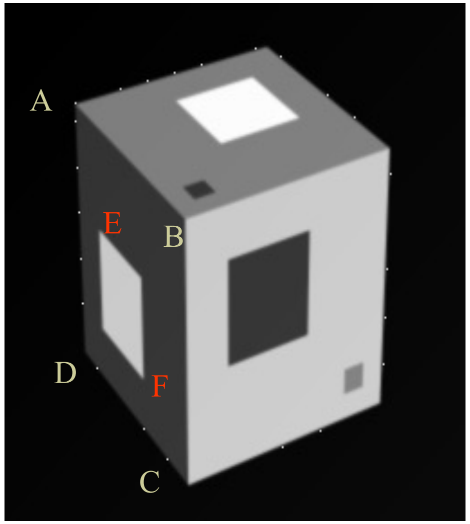
$$\mathbf{A}_{xy} \cdot \mathbf{h} = 0 \qquad \min_{\mathbf{h}} \left[\left(\mathbf{A}_{xy}\mathbf{h} \right)^2 \right]$$

$$\boxed{SVD(A_{xy}) \rightarrow A_{xy} = UDV^T \Rightarrow \mathbf{h} = v_n}$$

Homography between Points of a Plane

- Calculation of the Homography between Points on a Plane:
  - A set of matched points projections belonging to the plane must be found on both images
  - We obtain the matrix $\widetilde{H}_{\Pi}$
- It can be used to find correspondence from any point on the plane





Homography between Points of a Plane

- Example: Photogrammetry - 3D Modeling

Video

INRIA - Shopia Antipolis

Homography between Points (General Disposition)

$$\text{if} \quad \tilde{m} \cong \tilde{A}\ \tilde{M}_c = [A \quad 0]\ \tilde{M}_c \quad ; \qquad \tilde{m}' \cong \tilde{A}'\ \tilde{M}_{c'} = [A' \quad 0]\ \tilde{M}_{c'}$$

$$\tilde{M}_{c'} = \tilde{M}^{c}_{c'}\ \tilde{M}_c \quad \text{con} \quad \tilde{M}^{c}_{c'} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \ ; \ \tilde{M}_c{}^{c'} = \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}$$

- **Projecting in Camera 1:** $\tilde{M}_c = \begin{bmatrix} M_c \\ 1 \end{bmatrix} \in P^3 \quad \Rightarrow \quad \tilde{m} \cong A\ M_c \quad \Rightarrow \quad M_c \cong A^{-1}\tilde{m}$
- **Projecting in Camera 2:** $\tilde{M}_c \in P^3$ in the second camera $\tilde{m}' \cong \tilde{A}'\ \tilde{M}_{c'} = A' M_{c'}$

$$\text{Con} \quad \tilde{M}_{c'} = \begin{bmatrix} M_{c'} \\ 1 \end{bmatrix} = \tilde{M}^{c}_{c'} \begin{bmatrix} M_c \\ 1 \end{bmatrix} = \begin{bmatrix} RM_c + t \\ 1 \end{bmatrix} \quad \Rightarrow \quad M_{c'} = RM_c + t$$

$$\tilde{m}' \cong A' M_{c'} = A'(RM_c + t) = \lambda A' R A^{-1} \tilde{m} + A' t = H_\infty \tilde{m} + A' t$$

$$\boxed{\tilde{m}' \cong H_\infty \tilde{m} + \lambda\, \tilde{e}'} \qquad \boxed{\tilde{H}_\infty \cong A' R A^{-1}} \qquad \boxed{\tilde{e}' \cong A' \ t}$$

Epipolar Line

Homography between Points. Mosaic

- Mosaic: A sequence of images is acquired:
  - Without translation (t=0)
  - With rotation and change of intrinsic parameters

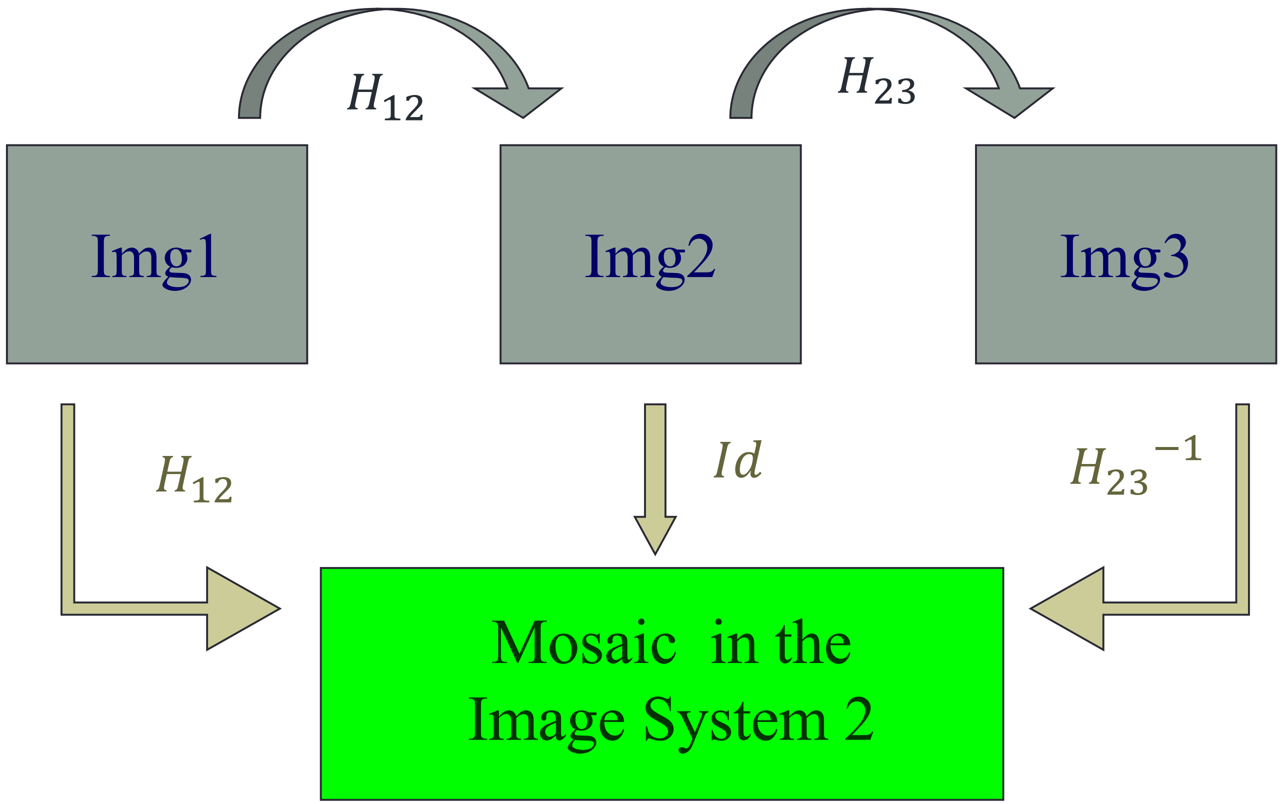
$$\text{if} \quad \tilde{m} \cong \tilde{A}\ \tilde{M}_c \cong [A \quad 0]\ \tilde{M}_c \cong A\ M_c \qquad \Rightarrow M_c \cong A^{-1}\tilde{m}$$

$$\tilde{m}' \cong \tilde{A}' \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tilde{M}_C \cong [A' \quad 0] \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \tilde{M}_C \cong A'R\ M_C$$

$$\tilde{m}' \cong A'RA^{-1}\tilde{m} \quad \Rightarrow \quad \tilde{m}' \cong \tilde{H}_\theta \tilde{m} \quad \text{con} \quad \tilde{H}_\theta = \tilde{H}_\infty \cong A'RA^{-1}$$

$$\boxed{\tilde{m}' \cong H_\theta\ \tilde{m} \quad \text{con} \quad H_\theta = H_\infty}$$

Homography between Points. Mosaic



Homography between Points. Mosaic

Video

Mosaic of the Carlton Hotel
(Imad Zoghlami)

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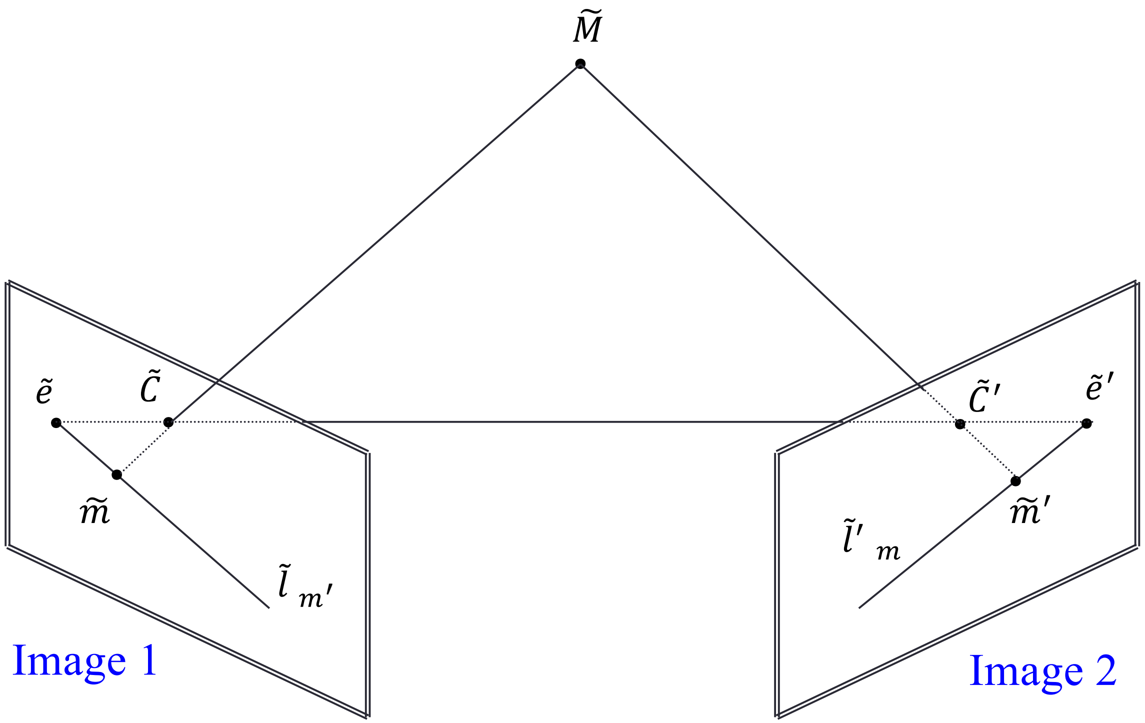
Epipolar Geometry

- Each point in one image is associated with an Epipolar line in the other image:

$$\tilde{m} \Rightarrow \tilde{l}'_m \qquad \tilde{m}' \Rightarrow \tilde{l}_{m'}$$



Epipolar Geometry

- Epipolar line in the second camera for a point on the first camera: $\tilde{m}$

$$\tilde{l}'_m \cong A'^{-T}[t]_{\wedge} RA^{-1}\tilde{m}$$

  - It can be expressed as:

$$\tilde{F} \cong A'^{-T}[t]_{\wedge} RA^{-1}$$

$$\tilde{l}'_m \cong \tilde{F}\,\tilde{m}$$

  - Transforms points in one image into lines in the other image (called correlation)
  - Similarly it can be obtained for a point of the second camera:

$$\tilde{l}_{m'} \cong A^{-T}R^T[t]_{\wedge} A'^{-1}\tilde{m}'$$

$$\tilde{l}_{m'} \cong \tilde{F}^T\,\tilde{m}'$$

Epipolar Geometry: Fundamental Matrix

- The matrix $\tilde{F}$ is called the Fundamental Matrix

$$\tilde{l}'_m \cong \tilde{F}\,\tilde{m} \qquad \tilde{l}_{m'} \cong \tilde{F}^T\,\tilde{m}'$$

- As: $\tilde{m}' \in \tilde{l}'_m \Rightarrow \tilde{m}'^T \cdot \tilde{l}'_m = 0 \Rightarrow \boxed{\tilde{m}'^T\,\tilde{F}\,\tilde{m} = 0}$
- As: $\tilde{m} \in \tilde{l}_{m'} \Rightarrow \tilde{m}^T \cdot \tilde{l}_{m'} = 0 \Rightarrow \boxed{\tilde{m}^T\,\tilde{F}^T\,\tilde{m}' = 0}$
- For the epipoles: $\boxed{\tilde{F} \cdot \tilde{e} = 0\ ; \qquad \tilde{F}^T \cdot \tilde{e}' = 0}$
- The Fundamental matrix summarizes all the correspondence information between two points originated by the projection of the same spatial point.

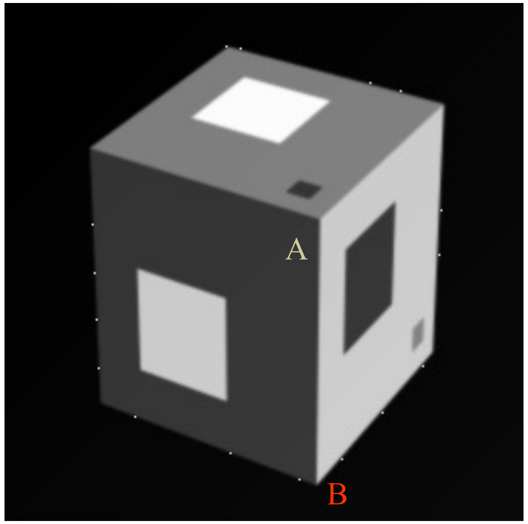
Epipolar Geometry: Fundamental Matrix

- Fundamental Matrix: $\tilde{m}'^{T} \; \tilde{F} \; \tilde{m} = 0$
  - Incorporates epipolar geometry information:
    - Depending solely on the correspondence between points in the two images
    - Without relying on knowledge of catchment parameters or the layout of the scene
  - The determinant of the fundamental matrix is 0, together with the scaling uncertainty gives SEVEN dof
  - For calculation, corresponding points are selected in the two images, and estimated by linear or nonlinear methods

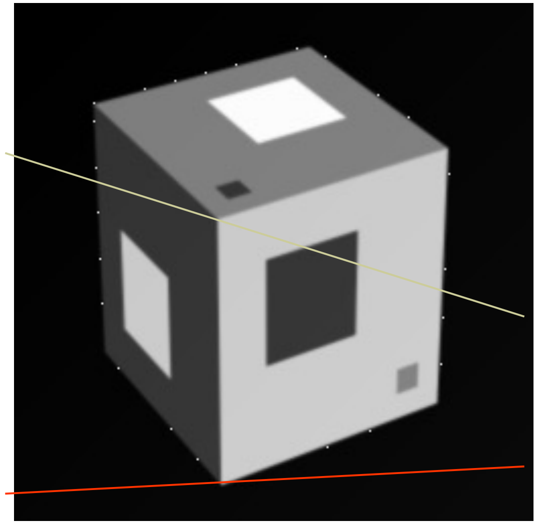
Epipolar Geometry: Fundamental Matrix

- For each point of the first (or second) image you can determine the epipolar line in the second (or first) image, such as the corresponding point will be on it.

$$\tilde{m}'^{T}\ \tilde{F}\ \tilde{m} = 0$$



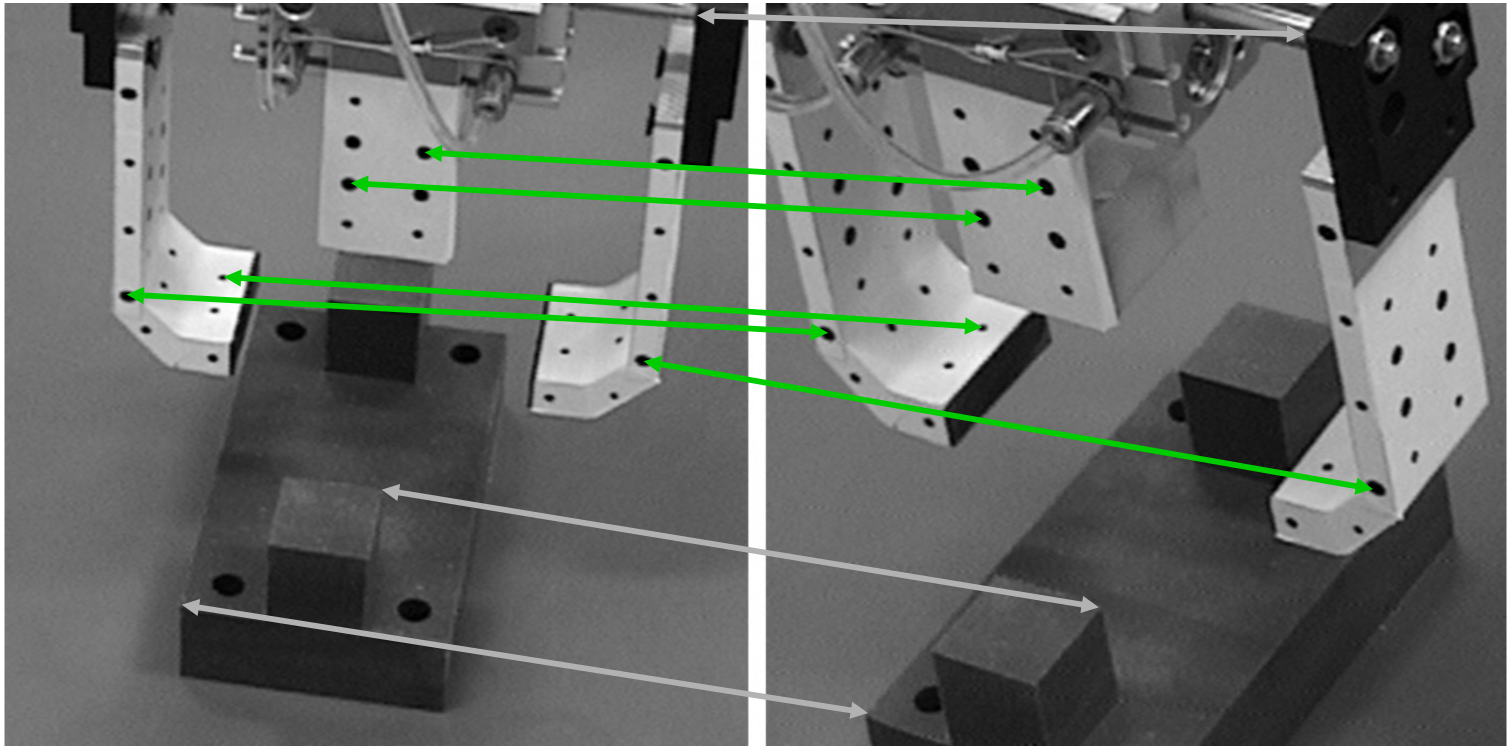
Left Camera



Right Camera

Epipolar Geometry: Fundamental Matrix

- Estimation of the Fundamental Matrix from a set of corresponding points in two images



- Easy correspondences determine the fundamental matrix (≥ 7)
- With the fundamental matrix we can found more difficult correspondences

Epipolar Geometry: Essential Matrix

- If you define the expression of a point in the image in Normalized coordinates as:

$$\tilde{m} \cong A\tilde{q} \;\Rightarrow\; \tilde{q} \cong A^{-1}\tilde{m} \;; \quad \tilde{m}' \cong A'\tilde{q}' \;\Rightarrow\; \tilde{q}' \cong A'^{-1}\tilde{m}'$$

$$\text{Then:} \quad \tilde{m}'^{T} F \tilde{m} \;=\; \tilde{m}'^{T} A'^{-T} [t]_{\wedge} R A^{-1} \tilde{m} = \tilde{q}'^{T} \; [t]_{\wedge} R \; \tilde{q} = 0$$

$$\boxed{\tilde{q}'^{T} \; E \; \tilde{q} = 0} \quad \text{with} \quad \boxed{E \cong [t]_{\wedge} R}$$

- If the coordinates are normalized, it implies that the intrinsic parameters are known
- Matrix ***E*** is called **Essential Matrix** and states that the Epipolar restriction depends only on the relative movement between the optical systems of the two cameras.
- It's especially useful when a mobile camera is being used ($A'=A$)

6 dof. - Two equal singular values

Epipolar Geometry: Essential Matrix

- The essential matrix can be applied to nonlinear camera models:

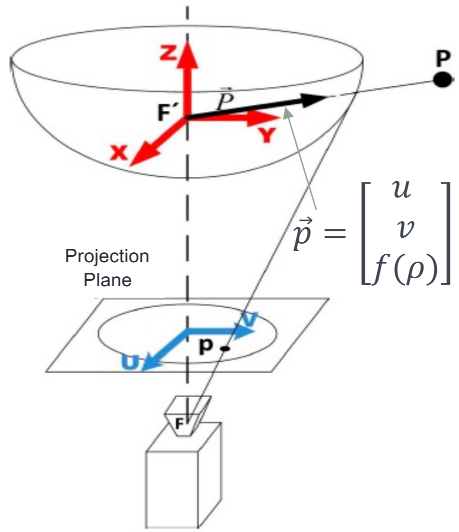
$$\tilde{q} \cong A^{-1}\ \tilde{m} = [I \quad 0]\ \widetilde{M}_c = \alpha \cdot M_c = \vec{p}$$

$$\tilde{q}' \cong A'^{-1}\ \tilde{m}' = [I \quad 0]\ \widetilde{M}_{c'} = \alpha \cdot M_{c'} = \vec{p}'$$

  - Where $\vec{p}$, $\vec{p}'$ are the projection rays from each camera

$$\vec{p}'^{T}\ E\ \vec{p} = 0 \qquad E \cong [t]_{\wedge}\ R$$

  - It is sufficient to have a model in which each image point is assigned a projection ray: (e.g. Omnidirectional Cameras)



$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} c & d \\ e & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

$$f(\rho) = a_0 + a_1\rho + \cdots + a_n\rho^n$$
$$\rho = \sqrt{u^2 + v^2}$$

Epipolar Geometry: Fundamental Matrix

- Summary:
  - The fundamental matrix has 7 degrees of freedom
    - 3 by the rotation
    - 2 by the translation (F doesn't depend on the magnitude of the translation, just the direction)
    - 2 by the intrinsic parameters (out of 10). The basis of projective calibration is to find the (nonlinear) relationships between them

$$F \cong A'^{-T}[t]_{\wedge}\, RA^{-1}$$

$$F \cong [e']_{\wedge}\, A'\, RA^{-1} \cong A'^{-T}\, RA^{T}[e]_{\wedge}$$

  - In the case of pure rotation, the fundamental matrix is null
  - Relationship between the fundamental matrix and the homography of any plane:

$$F^{T} H_{\pi} + H_{\pi}^{T} F = 0$$

$$F \cong [e']_{\wedge}\, H_{\pi}$$

Aligned Axes Cameras

- Aligned axes systems:
  - Epipoles at infinity

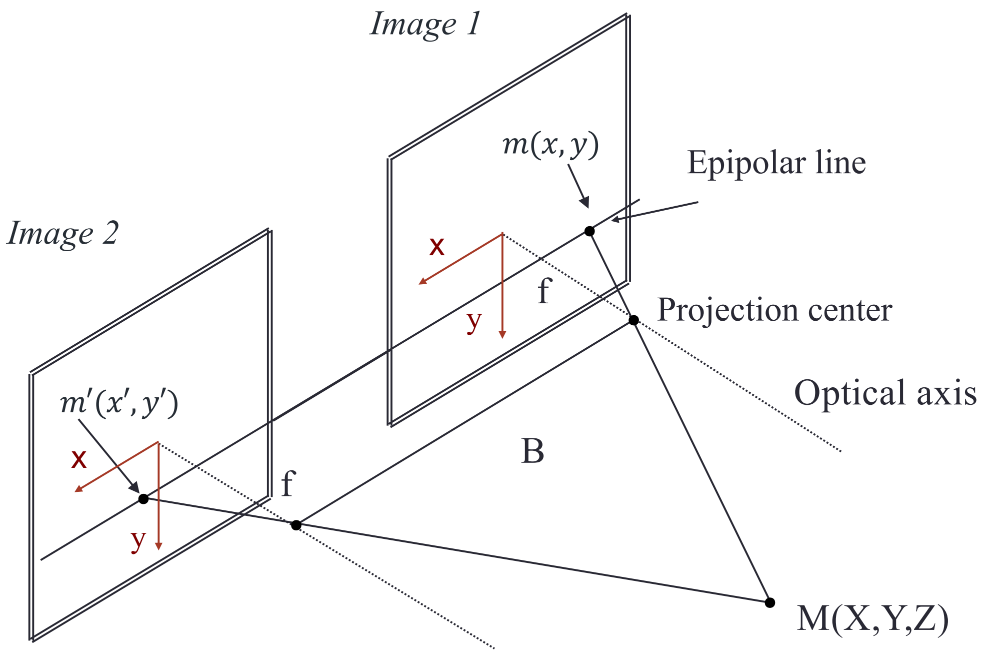
$$\tilde{F} \cong A'^{-T}[t]_\wedge RA^{-1}$$
$$F \cong [e']_\wedge A' RA^{-1} \cong A'^{-T} RA^{T}[e]_\wedge$$

$$\tilde{e} \cong A R^T t$$
$$\tilde{e}' \cong A' t$$

$$\boxed{\begin{aligned}\tilde{e} &\cong A t \\ \tilde{e}' &\cong A' t\end{aligned}}$$

$$\boxed{F \cong A'^{-T}[t]_\wedge A^{-1}} \xrightarrow[A = A']{} \boxed{F \cong [e]_\wedge \cong [e']_\wedge}$$

$$\tilde{e} \cong \tilde{e}' \cong \begin{bmatrix} \tilde{e}_x \\ 0 \\ 0 \end{bmatrix} \Rightarrow F \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$



Epipolar restriction:

$$\tilde{m}'^T \tilde{F} \tilde{m} = 0 \qquad \boxed{y = y'}$$

Disparity

$$\boxed{d = x' - x}$$

Comparison

- **ALIGNED AXES**
  - Advantage: Simplicity of calculation
  - Disadvantages:
    - Building difficulty
    - If B is high, the images are very different, with possible bad correspondences
    - If B is low, we get poor accuracy
- **CONVERGENT AXES**
  - Advantages:
    - Ease of construction
    - Simple correspondence if the disparity is low
  - Disadvantages:
    - Complejidad de cálculo
    - Si B es alto, correspondence is more difficult, occlusions
    - If B is low, we get poor accuracy

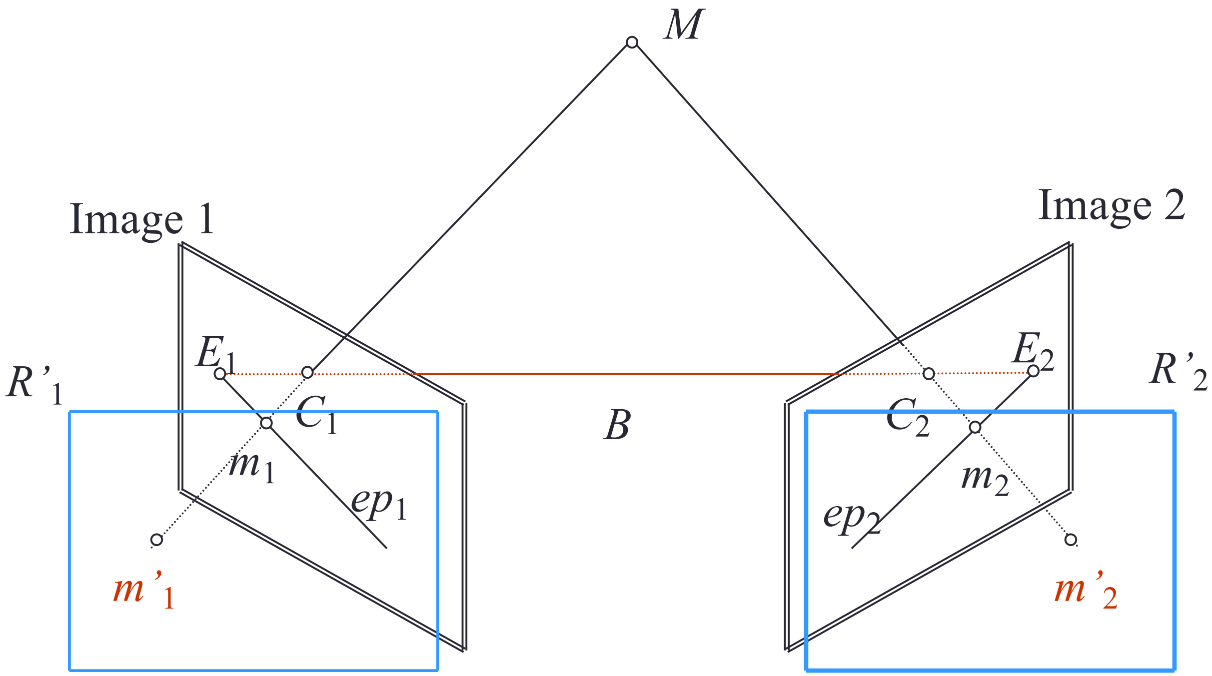
Rectification

- Rectify:
  - Reproject camera images into two new aligned projective planes

  - Advantage:
    - Simplify correspondence
  - Disadvantages:
    - Computational cost
    - Possible loss of accuracy

Rectification

- Rectify images using a Homography, to the desired disposition (aligned or convergent axes).

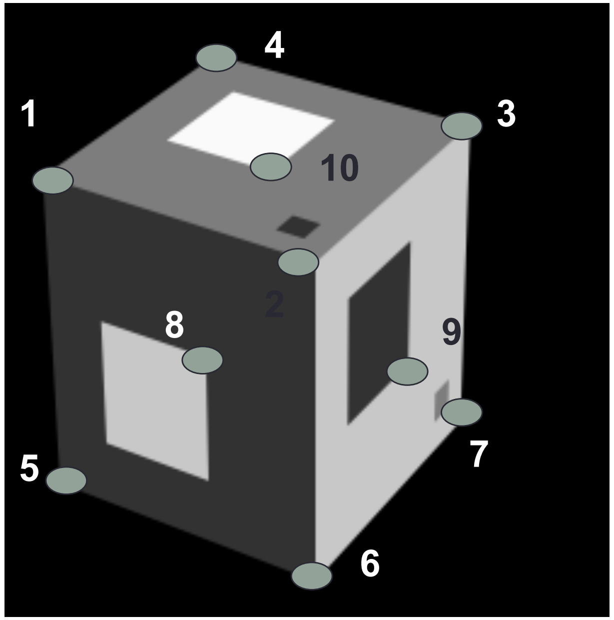


Rectification

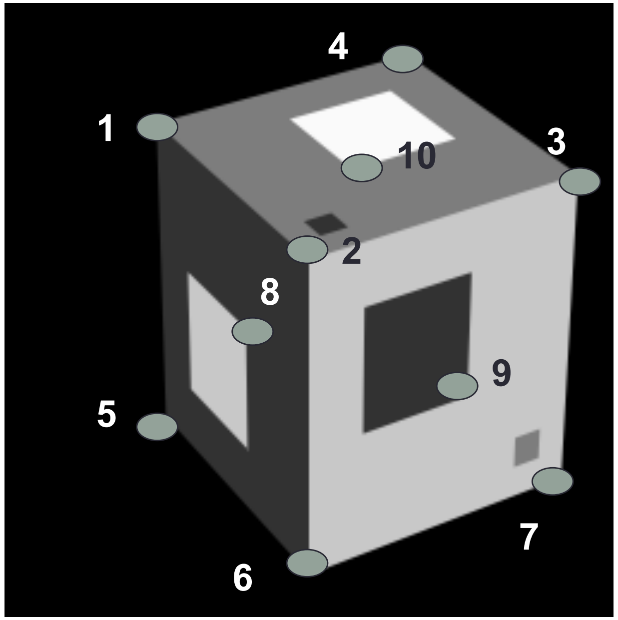
- Strategies:
  - All calibration data is known:
    - A direction of aligned axes is selected, calculating the angles
    - For each point of the original image you get the rectified point
  - Calibration data is not known exactly:
    - Corresponding points are obtained in both images
    - From them the homography is calculated for a desired projection (y=y')
    - For each point of the original image you get the rectified point

Rectification

- Example:



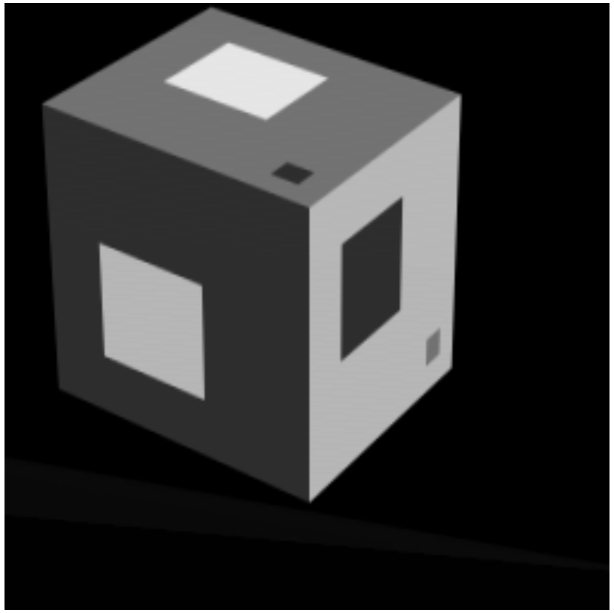
Left image



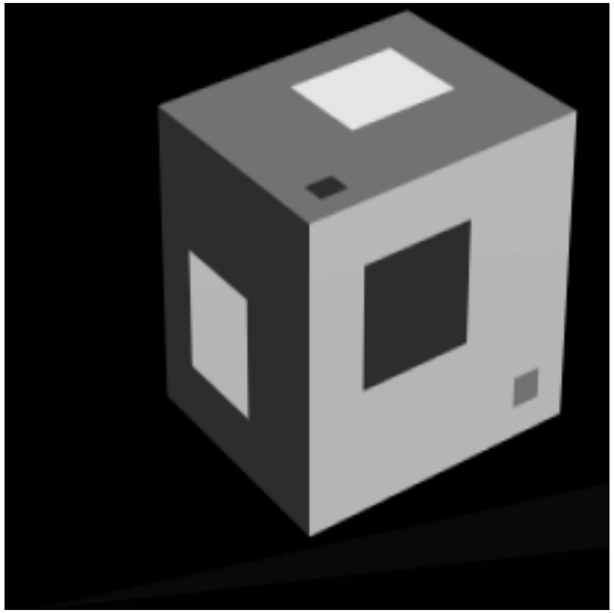
Right image

Rectification

- Example:

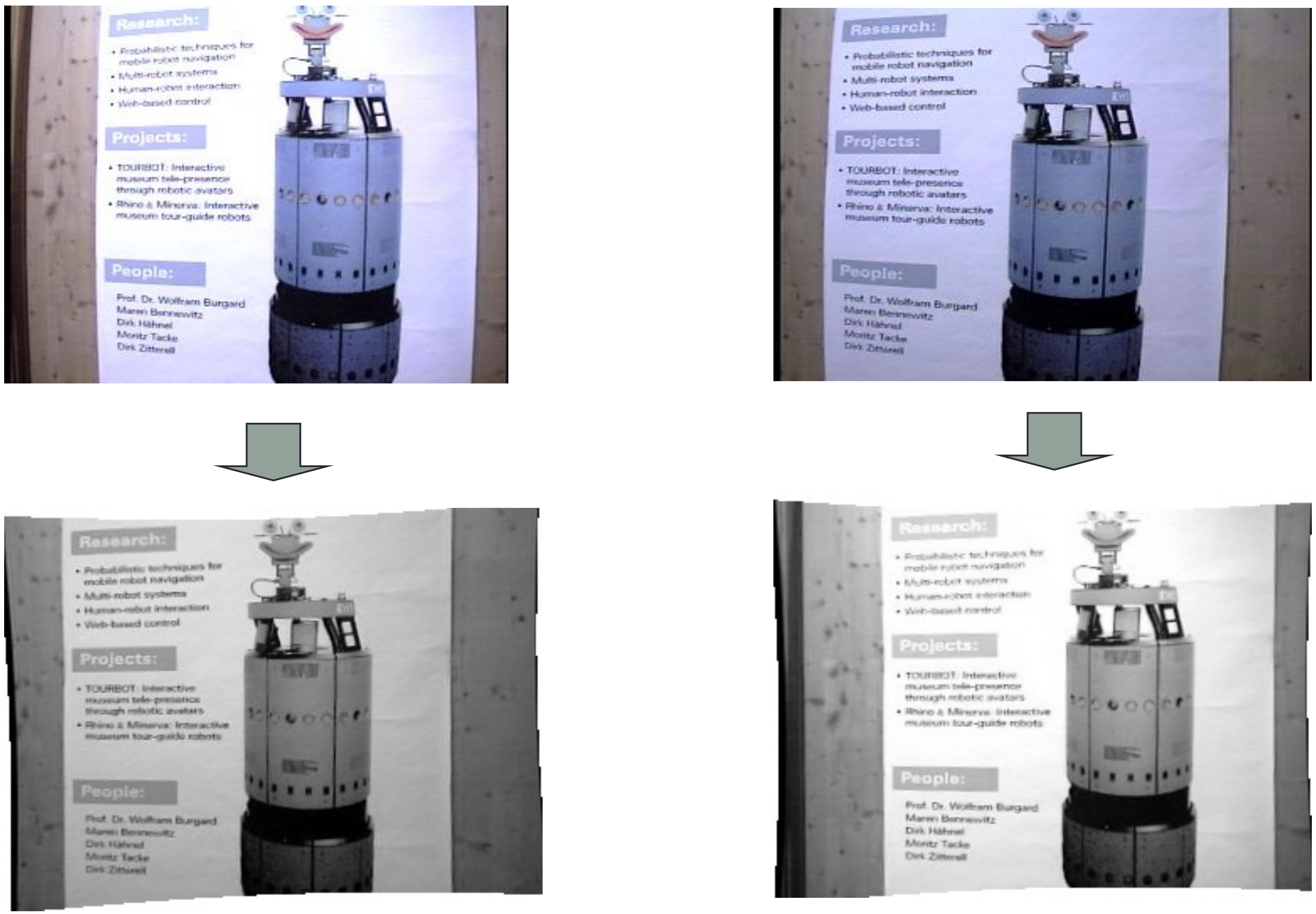


Rectified Left Image



Rectified Right Image

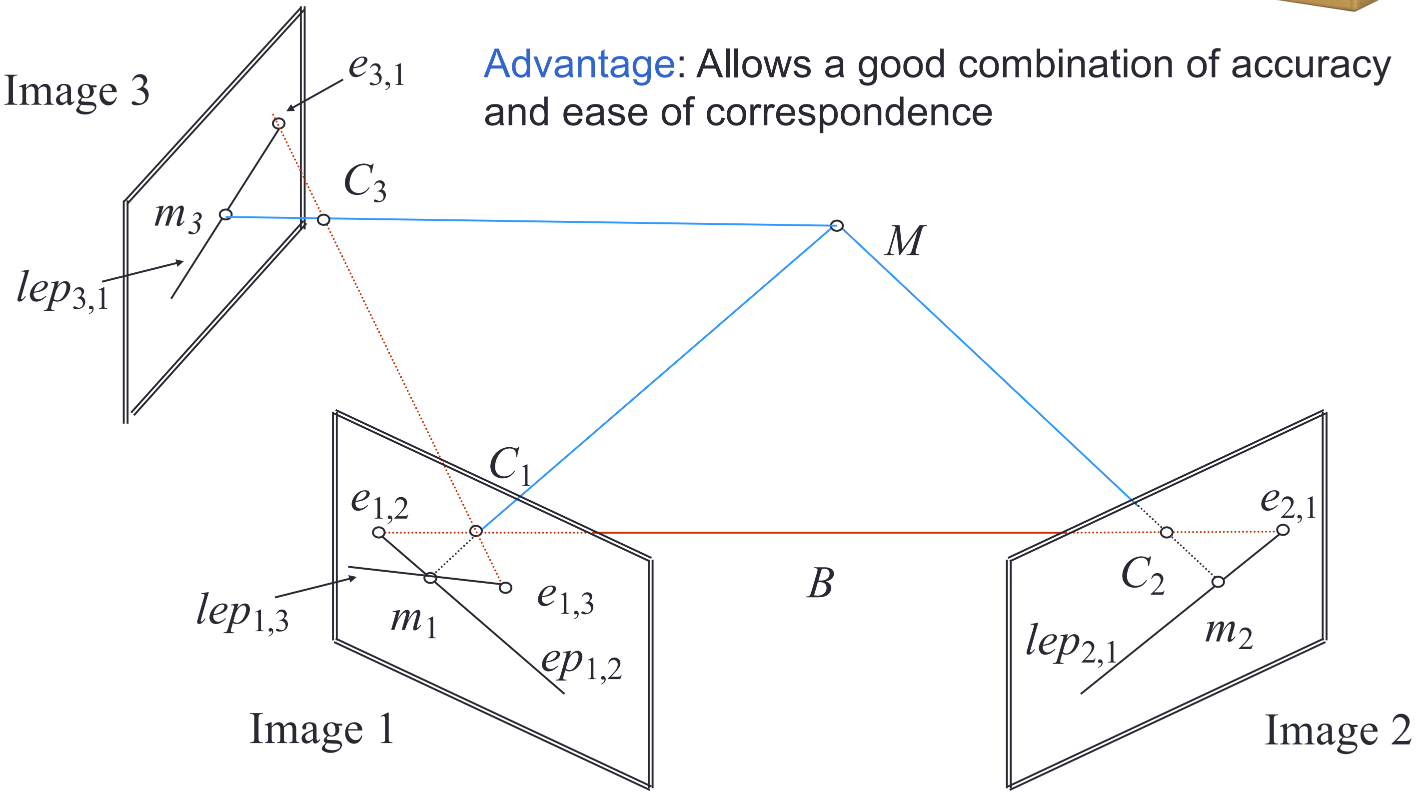
Rectification



Trinocular Vision: Epipolar Geometry

- Trifocal Tensor

Advantage: Allows a good combination of accuracy and ease of correspondence



Point Grey Bumblebee XB3 Trinocular Camera

Summary

- Projective Model of a Binocular Stereo Camera
- Homographies between different projective planes
- Next steps:
  - Projective Calibration → Fundamental Matrix / Essential Matrix
  - Correspondence (Matching)
  - 3D Reconstruction