3D VISION

Mathematical tools

Homography Estimation
Projective Calibration F/E
3D Triangulation

Systems Engineering and Automation

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Singular Values Decomposition (SVD)

- A is a square matrix: $A_{nxn} = U \cdot D \cdot V^T$
  - U, V are orthogonal matrices
  - D is a diagonal matrix, with non-negative values (singular values) sorted from highest to lowest

$$D_{nxn} = diag(\sigma_1, \dots, \sigma_n)$$

$$abs(|A|) = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$$

$$\mathrm{sgn}(|A|) = |U \cdot V^T|$$

- Matrix Condition
  - Optimal condition: $cond(A) \to 1$
  - Singular matrix: $cond(A) \to \infty$

$$cond(A) = \frac{\sigma_1}{\sigma_n}$$

- Relationship with eigen values:

$$A^T A = VDU^T \cdot UDV^T = VD^2V^T = VD^2V^{-1}$$

  - D^2 diagonal matrix with the eigen values of A^TA
  - V are the eigen vectors of A^TA
  - The singular values are the positive square roots of the eigen values of A^TA

Singular Values Decomposition (SVD)

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$$D_{nxn} = diag(\sigma_1, \dots, \sigma_n)$$

$$abs(|A|) = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$$

$$A_{nxn} = U \cdot \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \sigma_{n-1} & 0 \\ 0 & \cdots & 0 & \sigma_n \end{bmatrix} \cdot V^T$$

- Nearest singular matrix: $|A^*| = 0$ $cond(A^*) \to \infty$

$$A^*_{nxn} = U \cdot \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \sigma_{n-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} \cdot V^T$$

- Matrix orto-normalization: $|A^*| = 1$

$$A^*_{nxn} = [\bar{\mathbf{a}}^*_1 \quad \cdots \quad \bar{\mathbf{a}}^*_n]$$

$$\bar{\mathbf{a}}_i^{*T} \cdot \bar{\mathbf{a}}^*_j = \begin{cases} 0 & for \quad i \neq j \\ 1 & for \quad i = j \end{cases}$$

$$A^*_{nxn} = U \cdot \begin{bmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & \boxed{|U \cdot V^T|} \end{bmatrix} \cdot V^T$$

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Singular Values Decomposition (SVD)

- A is a non-square matrix:

$$A_{mxn} = U \cdot D \cdot V^T \qquad (m > n)$$

  - U_{mxm} is a matrix with orthogonal columns
  - D_{mxn} is a diagonal matrix (singular values)
  - V_{nxn} is an orthogonal matrix (singular vectors)

$$D_{mxn} = diag(\sigma_1, \dots, \sigma_n)$$

- Matrix Condition (to rank n)
  - Optimal condition: $cond(A) \rightarrow 1$
  - Singular matrix: $cond(A) \rightarrow \infty$

$$cond(A) = \frac{\sigma_1}{\sigma_n}$$

- Matrix Condition (to rank k)

$$cond(A) = \frac{\sigma_1}{\sigma_k}$$

Conditioning of linear equations systems

- A linear equations system is well conditioned if small changes in parameters produce small changes in the results

- **Condition number**: Ratio between major and lowest singular values
- Example of a well conditioned system $A\,x = b$

$$A = \begin{bmatrix} 8 & -5 \\ 4 & 10 \end{bmatrix} \rightarrow \begin{cases} b = \begin{bmatrix} 3 \\ 14 \end{bmatrix} & \Rightarrow\ x = A^{-1} \cdot b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ b = \begin{bmatrix} 3 - 0.04 \\ 14 - 0.06 \end{bmatrix} & \Rightarrow x = A^{-1} \cdot b = \begin{bmatrix} 0.993 \\ 0.996 \end{bmatrix} \end{cases}$$

$$vs = [11.1803 \quad 8.9443]$$

$$cond(A) = 1.25$$

- Example of an ill conditioned system $A\,x = b$

$$A = \begin{bmatrix} 0.66 & 3.34 \\ 1.99 & 10.01 \end{bmatrix} \rightarrow \begin{cases} b = \begin{bmatrix} 4 \\ 12 \end{bmatrix} & \Rightarrow x = A^{-1} \cdot b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ b = \begin{bmatrix} 4 - 0.04 \\ 12 - 0.06 \end{bmatrix} & \Rightarrow x = A^{-1} \cdot b = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \end{cases}$$

$$vs = [10.7588 \quad 0.0037]$$

$$cond(A) = 2.89 \cdot 10^3$$

Singular Values Decomposition (SVD)

- A is a non-square matrix :

$$A_{mxn} = U \cdot D \cdot V^T \qquad (m < n)$$

  - U_{mxm} is an matrix with orthogonal columns
  - D_{mxn} is a diagonal matrix (singular values) $\quad D_{mxn} = diag(\sigma_1, \dots, \sigma_m)$
  - V_{nxn} is an orthogonal matrix (singular vectors)

- Null space (kernel):
  - Solution of the equation. $A\,x=0$
    - r last non-null singular value (reasonable condition)
    - Solutions: $\lambda = n\text{-}r$ last singular vectors of V

$$[\mathbf{x}] = [v_{r+1} \quad \cdots \quad v_n]$$

Least Squares

- Linear Least Squares (Homogeneous Equation)
  - Solve $A\,x=0$ with A_{mxn}
  - Over-determined system $m \geq n$
    - Exact equation systems: ($\lambda \geq 1$ size of solution space)
      - (m-n+λ) restrictions should be linear combination of the remaining (n-λ)
      - λ null singular values
    - Systems of approximate equations: least squares solution
      - Minimize for x $\|e\|^2 = (Ax)^T(Ax)$
  - Solution: $SVD(A) \rightarrow A = UDV^T \Rightarrow \begin{cases} \lambda = 1 \rightarrow \mathbf{x} = v_n \\ \lambda > 1 \rightarrow [\mathbf{x}] = [v_{n-\lambda+1} \quad \cdots \quad v_n] \end{cases}$
  - Condition to rank (n-λ): $cond(A) = \dfrac{\sigma_1}{\sigma_{n-\lambda}}$

Least Squares

- Unrestricted linear least squares
- Solve $Ax=b$ with. A_{mxn} $m \geq n$
  - Solution: $x=(A^TA)^{-1}A^Tb$
  - Minimized with respect to: $||e||^2 = (Ax-b)^T(Ax-b)$
  - Generalized Problem: give a certain weight to each error
    - Minimize with respect to x: $(Ax-b)^T W^{-1}(Ax-b)$
    - Solution: $x=(A^T W^{-1} A)^{-1} A^T W^{-1} b$

- Example: closer 2D line to several points $(v=\boldsymbol{p}u+\boldsymbol{q})$

$$* \ \text{Points}(u,v)\left\{\begin{matrix}(1.3, \ -0.5)\\(1.7, \ 0.6)\\(2.4, \ 1.5)\end{matrix}\right\}; \ A=\begin{bmatrix}1.3 & 1.0\\1.7 & 1.0\\2.4 & 1.0\end{bmatrix}; \ b=\begin{bmatrix}-0.5\\0.6\\1.5\end{bmatrix} \rightarrow \begin{cases}\text{Solution: } (p,q)=(1.75,-2.63)\\\text{Error: } (0.1543,-0.2425, \ 0.0882)\\\text{Condition number: } 93.66\end{cases}$$

$$* \ \text{Points}(u,v)\left\{\begin{matrix}(1.3, \ -0.5)\\(1.7, \ 0.6)\\(2.4, \ 1.5)\end{matrix}\right\}; \ A=\begin{bmatrix}130.0 & 100.0\\1.7 & 1.0\\2.4 & 1.0\end{bmatrix}; \ b=\begin{bmatrix}-50.0\\0.6\\1.5\end{bmatrix} \rightarrow \begin{cases}\text{Solution: } (p,q)=(1.92,-3.00).\\\text{Error: } (0.0021,-0.3292, \ 0.1197)\\\text{Condition number:: } 52856\end{cases}$$

Data Normalization

- Better Conditioned Equations Systems :
  - Define a proyective transformation

    $\tilde{\tilde{m}} = \boldsymbol{T} \cdot \tilde{m}$ such that

    - The center of gravity of the points is the origin
    - The mean distance to origin is $\sqrt{2}$
    - Applies to image and space points

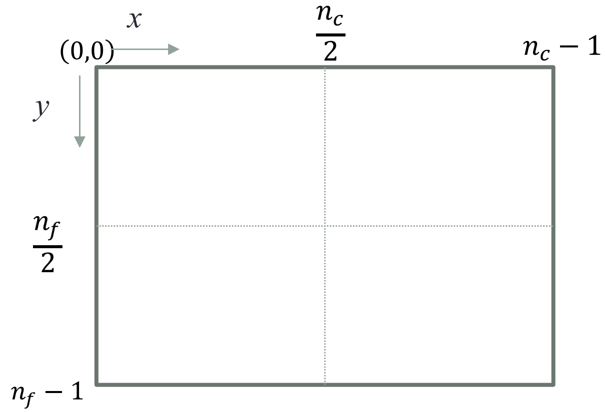
$$\boldsymbol{T} = \begin{bmatrix} 1/\sigma_x & 0 & -\mu_x/\sigma_x \\ 0 & 1/\sigma_y & -\mu_y/\sigma_y \\ 0 & 0 & 1 \end{bmatrix}$$

  - We can also employ simpler transformations
    - All the points between -1 y 1

$$\boldsymbol{T} = \begin{bmatrix} 2/n_c & 0 & -1 \\ 0 & 2/n_f & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

    - All the points between 0 y 2

$$\boldsymbol{T} = \begin{bmatrix} 2/n_c & 0 & 0 \\ 0 & 2/n_f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Nonlinear Optimization

- Levenberg-Marquardt nonlinear Optimization
  - Approach: Solve $\boldsymbol{a}=f(\boldsymbol{p})$
    - $\boldsymbol{a}$: Measures belonging to R^m
    - $\boldsymbol{p}$: Parameters belonging to R^n
  - Solution:
    - Find $\hat{p}$ that complies with $a = f(\hat{p}) + \varepsilon$ minimizing $\varepsilon = a - f(\hat{p})$
    - Linear approximation of $\boldsymbol{f}$ *around* $\boldsymbol{p}_i$: $a + \varepsilon_i = f(p_i + \Delta_i) = f(p_i) + J_i\Delta_i$

      con $J=\frac{\partial f}{\partial p}$; $J_i\Delta_i = \varepsilon_i \rightarrow \Delta_i = J_i{}^{+}\varepsilon_i = \left(J_i{}^T J_i\right)^{-1} J_i{}^T \varepsilon_i$ $J_i{}^{+}$ pseudo-inverse of J_i
    - Iterative process. $\boxed{p_{i+1} = p_i + \Delta_i}$
    - Acceleration: If $N = J_i{}^T J_i$
      - $N^{*}=(1+\lambda)N$ with initial value $\lambda=10^{-3}$
      - With $\|\varepsilon_{i+1}\| < \|\varepsilon_i\| \rightarrow \lambda=\lambda/10$
      - With $\|\varepsilon_{i+1}\| > \|\varepsilon_i\| \rightarrow \lambda=\lambda*10$

Matrix Decomposition

- **Cholesky** decomposition
  - If $\boldsymbol{S}$ is a symmetrical matrix of dimension nxn and $|\boldsymbol{S}|>0$ then exists a matrix $\boldsymbol{T}$ upper (or lower) triangular of the same dimension that meets:

$$\boldsymbol{S} = \boldsymbol{T}^T \boldsymbol{T}$$

  - If the elements of the matrix T are $\{t_{ij}\}$, with $t_{ij}=0$ for $i>j$

$$\boldsymbol{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ 0 & t_{22} & t_{23} \\ 0 & 0 & t_{33} \end{bmatrix} \qquad t_{ii} = +\sqrt{s_{ii} - \sum_{k=1}^{i-1} t_{ki}^2} \quad and \quad t_{ij} = \frac{1}{t_{ii}}\left(s_{ij} - \sum_{k=1}^{i-1} t_{ki} t_{kj}\right) \quad \text{for} \quad i < j$$

  - The formulation requires these elements to be calculated by rows, from left to right and from top to bottom

Homography Calculation

- **HOMOGRAPHY**: Linear relationship between points belonging to different projective spaces (e.g. 2 planes):

$$\boxed{\tilde{m}' \cong H \cdot \tilde{m}} \qquad \begin{bmatrix} n\cdot x' \\ n\cdot y' \\ n \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Longrightarrow \begin{matrix} (h_{31}x + h_{32}y + h_{33})\cdot x' = h_{11}x + h_{12}y + h_{13} \\ (h_{31}x + h_{32}y + h_{33})\cdot y' = h_{21}x + h_{22}y + h_{23} \end{matrix}$$

- Homogeneous Solution:

#points ≥ 4

$$\mathbf{A}_{xy} = \begin{bmatrix} -x^{(1)} & -y^{(1)} & -1 & 0 & 0 & 0 & x^{(1)}x'^{(1)} & y^{(1)}x'^{(1)} & x'^{(1)} \\ 0 & 0 & 0 & -x^{(1)} & -y^{(1)} & -1 & x^{(1)}y'^{(1)} & y^{(1)}y'^{(1)} & y'^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{A}_{xy}\cdot\mathbf{h} = 0 \qquad \mathbf{h} = [h_{11}, h_{12}, h_{13}, h_{21}, h_{22}, h_{23}, h_{31}, h_{32}, h_{33}]^T$$

$$\min_{\mathbf{h}} \left[(\mathbf{A}_{xy}\mathbf{h})^2 \right] \qquad \boxed{SVD(A_{xy}) \rightarrow A_{xy} = UDV^T \Rightarrow \mathbf{h} = v_9} \qquad cond(A_{xy}) = \frac{\sigma_1}{\sigma_8}$$

- Normalization: robust noise solution

$$\begin{matrix} \tilde{\tilde{m}} = T\cdot\tilde{m} \\ \tilde{\tilde{m}}' = T'\cdot\tilde{m}' \end{matrix} \longrightarrow \tilde{\tilde{m}}' \cong \hat{H}\cdot\tilde{\tilde{m}} \longrightarrow \boxed{H \cong T'^{-1}\cdot\hat{H}\cdot T}$$

Homography Calculation

- Non-homogeneous solution: assuming $(h_{33} \neq 0)$

$$\begin{bmatrix} n\cdot x' \\ n\cdot y' \\ n \end{bmatrix} = \begin{bmatrix} h_{11}/h_{33} & h_{12}/h_{33} & h_{13}/h_{33} \\ h_{21}/h_{33} & h_{22}/h_{33} & h_{23}/h_{33} \\ h_{31}/h_{33} & h_{32}/h_{33} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad h_{33} \cong t_z \quad \text{Distance between both planes}$$

$$\begin{bmatrix} n\cdot x' \\ n\cdot y' \\ n \end{bmatrix} = \begin{bmatrix} h'_{11} & h'_{12} & h'_{13} \\ h'_{21} & h'_{22} & h'_{23} \\ h'_{31} & h'_{32} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Longrightarrow \begin{array}{l} (h'_{31}x + h'_{32}y + 1)\cdot x' = h'_{11}x + h'_{12}y + h'_{13} \\ (h'_{31}x + h'_{32}y + 1)\cdot y' = h'_{21}x + h'_{22}y + h'_{23} \end{array}$$

$$\mathbf{A}_{xy} = \begin{bmatrix} -x^{(1)} & -y^{(1)} & -1 & 0 & 0 & 0 & x^{(1)}x'^{(1)} & y^{(1)}x'^{(1)} \\ 0 & 0 & 0 & -x^{(1)} & -y^{(1)} & -1 & x^{(1)}y'^{(1)} & y^{(1)}y'^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -x'^{(1)} \\ -y'^{(1)} \\ \vdots \end{bmatrix}^T \quad \#\text{points} \geq 4$$

$$\mathbf{h} = [h'_{11}, h'_{12}, h'_{13}, h'_{21}, h'_{22}, h'_{23}, h'_{31}, h'_{32}]^T \qquad \mathbf{A}_{xy}\cdot\mathbf{h} = \mathbf{b}$$

- Least Squares: $\boxed{\mathbf{h} = \left(A_{xy}^T A_{xy}\right)^{-1}\cdot A_{xy}^T\cdot\mathbf{b}}$ $\quad \min_{\mathbf{h}} \left[\left(\mathbf{A}_{xy}\mathbf{h} - \mathbf{b}\right)^2\right] \quad cond\left(A_{xy}\right) = \frac{\sigma_1}{\sigma_8}$

- Normalization: robust noise solution

$$\left.\begin{array}{l} \tilde{\tilde{m}} = T\cdot\tilde{m} \\ \tilde{\tilde{m}}' = T'\cdot\tilde{m}' \end{array}\right\} \longrightarrow \tilde{\tilde{m}}' \cong \hat{H}\cdot\tilde{\tilde{m}} \longrightarrow \boxed{H \cong T'^{-1}\cdot\hat{H}\cdot T}$$

Fundamental/Essential Matrix Calculation

- **Correlation**: Linear relationship between points and lines belonging to different projective spaces (e.g. 2 image planes)

$$\boxed{\tilde{m}'^{T}_{i} F \, \tilde{m}_i = 0} \qquad \tilde{m}' = [x' \quad y' \quad 1]^T, \quad \tilde{m} = [x \quad y \quad 1]^T \qquad F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$\mathrm{x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + f_{33} = 0}$$

- Homogeneous Solution: $\#\text{points} \geq 8$

$$\begin{bmatrix} \mathrm{x'_1x_1} & \mathrm{x'_1y_1} & \mathrm{x'_1} & \mathrm{y'_1x_1} & \mathrm{y'_1y_1} & \mathrm{y'_1} & \mathrm{x_1} & \mathrm{y_1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathrm{x'_nx_n} & \mathrm{x'_ny_n} & \mathrm{x'_n} & \mathrm{y'_nx_n} & \mathrm{y'_ny_n} & \mathrm{y'_n} & \mathrm{x_n} & \mathrm{y_n} & 1 \end{bmatrix} \mathbf{f} = 0 \qquad \boxed{A_{xy} \, \mathbf{f} = 0} \qquad \min_{\mathbf{f}} \left[\left(\mathbf{A}_{xy} \mathbf{f} \right)^2 \right]$$

$$\mathbf{f} = [f_{11} \quad f_{12} \quad f_{13} \quad f_{21} \quad f_{22} \quad f_{23} \quad f_{31} \quad f_{32} \quad f_{33}]^T$$

$$SVD(A_{xy}) \rightarrow A_{xy} = UDV^T \Rightarrow \mathbf{f} = v_9$$

$$\|\mathbf{f}\| = 1$$

$$\boxed{\mathbf{f} \;\Rightarrow\; F^*}$$

- Restrictions: F must be singular $|F| = 0$

$$SVD(F^*) \rightarrow F^* = UDV^T$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \rightarrow D^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{F = UD^*V^T} \qquad |F| = 0$$

Fundamental/Essential Matrix Calculation

- Correlation:
  - Non-Homogeneous Solution: assuming $(f_{33} \neq 0) \Rightarrow f_{33} = 1$

$$x'xf_{11} + x'yf_{12} + x'f_{13} + y'xf_{21} + y'yf_{22} + y'f_{23} + xf_{31} + yf_{32} + 1 = 0$$

$$\begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n \end{bmatrix} \mathbf{f} = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}$$

$\#\text{points} \geq 8$

$$\mathbf{f} = \begin{bmatrix} f_{11} & f_{12} & f_{31} & f_{21} & f_{22} & f_{23} & f_{31} & f_{32} \end{bmatrix}^T$$

$$\boxed{A_{xy}\ \mathbf{f} = \mathbf{b}} \qquad \min_{\mathbf{f}} \left[\left(A_{xy}\mathbf{f} - \mathbf{b} \right)^2 \right]$$

$$\mathbf{f} = \left(A_{xy}^T A_{xy} \right)^{-1} \cdot A_{xy}^T \cdot \mathbf{b} \qquad \boxed{\mathbf{f} \Rightarrow F^*}$$

- Restrictions: F must be singular $|F| = 0$

$$SVD(F^*) \rightarrow F^* = UDV^T$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \rightarrow D^* = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{F = UD^*V^T} \qquad |F| = 0$$

Fundamental/Essential Matrix Calculation

- Normalization

$$\left.\begin{aligned}\tilde{\tilde{m}} &= T \cdot \tilde{m} \\ \tilde{\tilde{m}}' &= T' \cdot \tilde{m}'\end{aligned}\right\} \longrightarrow \quad \tilde{\tilde{m}}'^{T}_{i} \hat{F}\ \tilde{\tilde{m}}_i = 0 \qquad \boxed{F = T'\hat{F}\ T}$$

$$T^{-1} = T'^{-1} = \begin{bmatrix} n_c/2 & 0 & n_c/2 \\ 0 & n_f/2 & n_f/2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Essential Matrix:
  - Additional restriction: two equal singular values

$$\tilde{q}'_i E\ \tilde{q}_i = 0 \qquad SVD(E*) \rightarrow E *= UDV^T \qquad E = UD^*V^T, \qquad |E| = 0$$

$$\tilde{q}_i = A^{-1}\,\tilde{m}$$

$$\tilde{q}'_i = A'^{-1}\,\tilde{m}'$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \rightarrow D^* = \begin{bmatrix} {(\lambda_1+\lambda_2)}/{2} & 0 & 0 \\ 0 & {(\lambda_1+\lambda_2)}/{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Fundamental/Essential Matrix Calculation

- 7-point Algorithm: $n=7$

$$\begin{bmatrix} x'_1x_1 & x'_1y_1 & x'_1 & y'_1x_1 & y'_1y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_nx_n & x'_ny_n & x'_n & y'_nx_n & y'_ny_n & y'_n & x_n & y_n & 1 \end{bmatrix} \mathbf{f} = 0 \qquad \boxed{A_{xy}\ \mathbf{f} = 0}$$

$$\mathbf{f} = [f_{11} \quad f_{12} \quad f_{13} \quad f_{21} \quad f_{22} \quad f_{23} \quad f_{31} \quad f_{32} \quad f_{33}]^T$$

- The null space of A_{xy} will be of dimension 2 (f_1, f_2)->(F_1, F_2)

$$[\mathbf{f}_1 \quad \mathbf{f}_2] = [v_8 \quad v_9] \qquad F = \alpha F_1 + (1-\alpha)F_2$$

- Restrictions: F singular

$$\det(\alpha F_1 + (1-\alpha)F_2) = 0$$

  - Analytical solution: cubic equation in α
  - Iterative solution: nonlinear minimization

- The normalization and denormalization process indicated previously should be applied

Triangulación

- Homogeneous Solution
  - Projection Rays: $\tilde{P} \cong A\ [I \quad 0]$ $\tilde{P}' \cong A'[R \quad t]$

$$\tilde{m}_i \cong \tilde{P}\ \tilde{M}_i \Rightarrow \begin{bmatrix} \lambda\ x \\ \lambda\ y \\ \lambda \end{bmatrix} = \begin{bmatrix} P_1^T \\ P_2^T \\ P_3^T \end{bmatrix} \tilde{M}_i \qquad \tilde{m}'_i \cong \tilde{P}'\ \tilde{M}_i \Rightarrow \begin{bmatrix} \mu x' \\ \mu\ y' \\ \mu \end{bmatrix} = \begin{bmatrix} P'^T_1 \\ P'^T_2 \\ P'^T_3 \end{bmatrix} \tilde{M}_i$$

$$\begin{matrix} P_3^T \tilde{M}_i x = P_1^T\ \tilde{M}_i \\ P_3^T \tilde{M}_i y = P_2^T\ \tilde{M}_i \end{matrix} \Rightarrow \begin{bmatrix} P_3^T x - P_1^T \\ P_3^T y - P_2^T \end{bmatrix} \tilde{M}_i = 0 \qquad \begin{matrix} P'^T_3 \tilde{M}_i x' = P'^T_1\ \tilde{M}_i \\ P'^T_3 \tilde{M}_i y' = P'^T_2\ \tilde{M}_i \end{matrix} \Rightarrow \begin{bmatrix} P'^T_3 x - P'^T_1 \\ P'^T_3 y - P'^T_2 \end{bmatrix} \tilde{M}_i = 0$$

$$\begin{bmatrix} P_3^T x - P_1^T \\ P_3^T y - P_2^T \\ P'^T_3 x' - P'^T_1 \\ P'^T_3 y' - P'^T_2 \end{bmatrix} \tilde{M}_i = 0 \qquad \boxed{A_{xy}\ \tilde{M}_i = 0}$$

- Least Squares estimation: $\min_{\tilde{M}} \left[\left(\left| A_{xy} \tilde{M} \right| \right)^2 \right]$

$$SVD(A_{xy}) \rightarrow A_{xy} = UDV^T$$
$$\tilde{M} = v_n \qquad \left\| \tilde{M} \right\| = 1$$

Triangulation

- Non-homogeneous solution - Euclidean:

$$\begin{cases}(p_{31}x - p_{11})X_w + (p_{32}x - p_{12})Y_w + (p_{33}x - p_{13})Z_w = p_{14} - p_{34}x \\ (p_{31}y - p_{21})X_w + (p_{32}y - p_{22})Y_w + (p_{33}y - p_{23})Z_w = p_{24} - p_{34}y\end{cases}$$
$$\begin{cases}(p'_{31}x' - p'_{11})X_w + (p'_{32}x' - p'_{12})Y_w + (p'_{33}x' - p'_{13})Z_w = p'_{14} - p'_{34}x' \\ (p'_{31}y' - p'_{21})X_w + (p'_{32}y' - p'_{22})Y_w + (p'_{33}y' - p'_{23})Z_w = p'_{24} - p'_{34}y'\end{cases}$$

$$A_{xy}M = b$$

$$A_{xy} = \begin{bmatrix} (p_{31}x - p_{11}) & (p_{32}x - p_{12}) & (p_{33}x - p_{13}) \\ (p_{31}y - p_{21}) & (p_{32}y - p_{22}) & (p_{33}y - p_{23}) \\ (p'_{31}x' - p'_{11}) & (p'_{32}x' - p'_{12}) & (p'_{33}x' - p'_{13}) \\ (p'_{31}y' - p'_{21}) & (p'_{32}y' - p'_{22}) & (p'_{33}y' - p'_{23}) \end{bmatrix} \quad b = \begin{bmatrix} p_{14} - p_{34} \\ p_{24} - p_{34} \\ p'_{14} - p'_{34} \\ p'_{24} - p'_{34} \end{bmatrix}$$

- Least squares estimation (pseudo-inverse): $\min_M \left[(A_{xy}M - b)^2\right]$

$$\boxed{M = \left(A_{xy}^T A_{xy}\right)^{-1} A_{xy}^T \cdot b}$$