# 3D VISION

## Calibration

DLT

Tsai

Systems Engineering and Automation

Universidad Miguel Hernández

# Direct Linear Transformation Method

- Direct Linear Transformation (DLT)
  - The calibration is done in two steps.
    - First, the elements of the Projection Matrix are calculated
      - Without imposing physical restrictions
    - From the previous estimation, intrinsic and extrinsic parameters are obtained
      - Imposing physical restrictions on matrix P

  - Calculates intrinsic and extrinsic parameters except for distortion.
  - Calculation of distortion (if modeled) at a later step
  - Calculation of other systematic errors.

$$\begin{bmatrix} n\,x_f \\ n\,y_f \\ n \end{bmatrix} = \tilde{P} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}}_1 & p_{14} \\ \bar{\mathbf{p}}_2 & p_{24} \\ \bar{\mathbf{p}}_3 & p_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

# Direct Linear Transformation Method

- Calculation of the Projection Matrix without considering the restrictions: non-homogeneous solution

$$\frac{1}{p_{34}}\tilde{P} = \begin{bmatrix} p_{11}/p_{34} & p_{12}/p_{34} & p_{13}/p_{34} & p_{14}/p_{34} \\ p_{21}/p_{34} & p_{22}/p_{34} & p_{23}/p_{34} & p_{24}/p_{34} \\ p_{31}/p_{34} & p_{31}/p_{34} & p_{31}/p_{34} & 1 \end{bmatrix} \Rightarrow \tilde{L} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & 1 \end{bmatrix}$$

$$\begin{bmatrix} n\,x_f \\ n\,y_f \\ n \end{bmatrix} = \begin{bmatrix} fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3 & fK_xt_x + C_xt_z \\ fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3 & fK_yt_y + C_yt_z \\ \bar{\mathbf{r}}_3 & t_z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\boxed{t_z = p_{34}}$$

$$x_f = \frac{X_wL_1 + Y_wL_2 + Z_wL_3 + L_4}{X_wL_9 + Y_wL_{10} + Z_wL_{11} + 1}$$

$$y_f = \frac{X_wL_5 + Y_wL_6 + Z_wL_7 + L_8}{X_wL_9 + Y_wL_{10} + Z_wL_{11} + 1}$$

$$x_f \cdot X_wL_9 + x_f \cdot Y_wL_{10} + x_f \cdot Z_wL_{11} + x_f = X_wL_1 + Y_wL_2 + Z_wL_3 + L_4$$
$$y_f \cdot X_wL_9 + y_f \cdot Y_wL_{10} + y_f \cdot Z_wL_{11} + y_f = X_wL_5 + Y_wL_6 + Z_wL_7 + L_8$$

# Direct Linear Transformation Method

- Calculation of the Projection Matrix without considering the restrictions.
- With *n* points (known $X^i_w$, $Y^i_w$, $Z^i_w$, $x^i_f$, $x^i_f$) we get a system with *11* unknowns and *2n* equations
- Least squares solution: $X_w L_1 + Y_w L_2 + Z_w L_3 + L_4 - x_f \cdot X_w L_9 - x_f \cdot Y_w L_{10} - x_f \cdot Z_w L_{11} = x_f$
  $X_w L_5 + Y_w L_6 + Z_w L_7 + L_8 - y_f \cdot X_w L_9 - y_f \cdot Y_w L_{10} - y_f \cdot Z_w L_{11} = y_f$

$$\begin{pmatrix} X^i_w & Y^i_w & Z^i_w & 1 & 0 & 0 & 0 & 0 & -X^i_w x^i_f & -Y^i_w x^i_f & -Z^i_w x^i_f \\ 0 & 0 & 0 & 0 & X^i_w & Y^i_w & Z^i_w & 1 & -X^i_w y^i_f & -Y^i_w y^i_f & -Z^i_w y^i_f \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & & & & & \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_7 \\ L_8 \\ L_9 \\ L_{10} \\ L_{11} \end{pmatrix} = \begin{pmatrix} x^i_f \\ y^i_f \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

$$\text{Sistema} \quad A_{xy}\mathbf{l} = \mathbf{b}$$

$$\mathbf{l} = \left(A^T_{xy} A_{xy}\right)^{-1} A^T_{xy} \cdot \mathbf{b}$$

# Direct Linear Transformation Method

- Projection Matrix Restrictions:

$$\begin{bmatrix} n\,x_f \\ n\,y_f \\ n \end{bmatrix} = \begin{bmatrix} fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3 & fK_xt_x + C_xt_z \\ fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3 & fK_yt_y + C_yt_z \\ \bar{\mathbf{r}}_3 & t_z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} n\,x_f \\ n\,y_f \\ n \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{p}}_1 & p_{14} \\ \overline{\mathbf{p}}_2 & p_{24} \\ \overline{\mathbf{p}}_3 & p_{34} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$Restrictions$:

$\|\overline{\mathbf{p}}_3\| = 1$

$(\overline{\mathbf{p}}_1 \wedge \overline{\mathbf{p}}_3) \cdot (\overline{\mathbf{p}}_2 \wedge \overline{\mathbf{p}}_3) = 0$

# Direct Linear Transformation Method

- Using the ***L*** parameters, and considering the physical meaning of the matrix ***P***, its elements are obtained.

$$\begin{bmatrix} n\,x_f \\ n\,y_f \\ n \end{bmatrix} = \begin{bmatrix} fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3 & fK_xt_x + C_xt_z \\ fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3 & fK_yt_y + C_yt_z \\ \bar{\mathbf{r}}_3 & t_z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} \qquad \frac{1}{p_{34}}\tilde{P} = \begin{bmatrix} p_{11}/p_{34} & p_{12}/p_{34} & p_{13}/p_{34} & p_{14}/p_{34} \\ p_{21}/p_{34} & p_{22}/p_{34} & p_{23}/p_{34} & p_{24}/p_{34} \\ p_{31}/p_{34} & p_{31}/p_{34} & p_{31}/p_{34} & 1 \end{bmatrix} \Rightarrow \tilde{L} = \frac{1}{p_{34}}\tilde{P} = \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & 1 \end{bmatrix}$$

$$\boxed{\|\bar{\mathbf{p}}_3\| = \|\bar{r}_3\| = 1} \Longrightarrow \boxed{t_z = p_{34} = \frac{1}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}}}$$

$$\tilde{P} = \begin{bmatrix} \frac{L_1}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_2}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_3}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_4}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} \\ \frac{L_5}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_6}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_7}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_8}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} \\ \frac{L_9}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_{10}}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{L_{11}}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} & \frac{1}{\sqrt{L_9^2 + L_{10}^2 + L_{11}^2}} \end{bmatrix}$$

# Direct Linear Transformation Method

- Calculation of intrinsic and extrinsic parameters.

$$\tilde{P} = \begin{bmatrix} fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3 & fK_xt_x + C_xt_z \\ fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3 & fK_yt_y + C_yt_z \\ \bar{\mathbf{r}}_3 & t_z \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{p}}_1 & p_{14} \\ \bar{\mathbf{p}}_2 & p_{24} \\ \bar{\mathbf{p}}_3 & p_{34} \end{bmatrix}$$

- Immediate calculation of $\bar{\mathbf{r}}_3$ and $t_z$
- Coordinates of the Central Point

$$\bar{\mathbf{p}}_1 \cdot \bar{\mathbf{p}}_3^T = (fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3) \cdot \bar{\mathbf{r}}_3{}^T = fK_x\bar{\mathbf{r}}_1 \cdot \bar{\mathbf{r}}_3{}^T + C_x\bar{\mathbf{r}}_3 \cdot \bar{\mathbf{r}}_3{}^T = C_x$$
$$\bar{\mathbf{p}}_2 \cdot \bar{\mathbf{p}}_3^T = \left(fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3\right) \cdot \bar{\mathbf{r}}_3{}^T = fK_y\bar{\mathbf{r}}_2 \cdot \bar{\mathbf{r}}_3{}^T + C_y\bar{\mathbf{r}}_3 \cdot \bar{\mathbf{r}}_3{}^T = C_y$$

# Direct Linear Transformation Method

- Calculation of intrinsic and extrinsic parameters:
  - Calculation of focal lengths in x and y

$$(*)\quad \bar{\mathbf{p}}_1 \cdot \bar{\mathbf{p}}_1^T - [\bar{\mathbf{p}}_1 \cdot \bar{\mathbf{p}}_3^T]^2 = (fK_x\bar{r}_1 + C_x\bar{r}_3)\cdot(fK_x\bar{r}_1 + C_x\bar{r}_3)^T - \left[(fK_x\bar{r}_1 + C_x\bar{r}_3)\cdot\bar{r}_3{}^T\right]^2 =$$

$$fK_x\bar{r}_1 \cdot fK_x\bar{r}_1{}^T + C_x\bar{r}_3 \cdot fK_x\bar{r}_1{}^T + fK_x\bar{r}_1 \cdot C_x\bar{r}_3{}^T + C_x\bar{r}_3 \cdot C_x\bar{r}_3{}^T - \left[fK_x\bar{r}_1 \cdot \bar{r}_3{}^T - C_x\bar{r}_3 \cdot \bar{r}_3{}^T\right]^2 =$$

$$\boxed{= f^2K_x^2 + C_x^2 - C_x^2 = f^2K_x^2 = f_x^2 \quad \Rightarrow \quad f_x}$$

$$(*)\quad \bar{\mathbf{p}}_2 \cdot \bar{\mathbf{p}}_2^T - [\bar{\mathbf{p}}_2 \cdot \bar{\mathbf{p}}_3^T]^2 = \left(fK_y\bar{r}_2 + C_y\bar{r}_3\right)\cdot\left(fK_y\bar{r}_2 + C_y\bar{r}_3\right)^T - \left[\left(fK_y\bar{r}_2 + C_y\bar{r}_3\right)\cdot\bar{r}_3{}^T\right]^2 =$$

$$fK_y\bar{r}_2 \cdot fK_y\bar{r}_2{}^T + C_y\bar{r}_3 \cdot fK_y\bar{r}_2{}^T + fK_y\bar{r}_2 \cdot C_y\bar{r}_3{}^T + C_y\bar{r}_3 \cdot C_y\bar{r}_3{}^T - \left[fK_y\bar{r}_2 \cdot \bar{r}_3{}^T - C_y\bar{r}_3 \cdot \bar{r}_3{}^T\right]^2 =$$

$$\boxed{= f^2K_y^2 + C_y^2 - C_y^2 = f^2K_y^2 = f_y^2 \quad \Rightarrow \quad f_y}$$

# Direct Linear Transformation Method

- Calculation of intrinsic and extrinsic parameters.
  - Calculation of remaining extrinsic parameters.

$$\bar{\mathbf{p}}_1 = fK_x\bar{\mathbf{r}}_1 + C_x\bar{\mathbf{r}}_3 \quad \Rightarrow \quad \bar{\mathbf{r}}_1 = \frac{\bar{\mathbf{p}}_1 - C_x\bar{\mathbf{r}}_3}{fK_x}$$

$$\bar{\mathbf{p}}_2 = fK_y\bar{\mathbf{r}}_2 + C_y\bar{\mathbf{r}}_3 \quad \Rightarrow \quad \bar{\mathbf{r}}_2 = \frac{\bar{\mathbf{p}}_2 - C_y\bar{\mathbf{r}}_3}{fK_y}$$

$$p_{14} = fK_xt_x + C_xt_z \quad \Rightarrow \quad t_x = \frac{p_{14} - C_xt_z}{fK_x}$$

$$p_{24} = fK_yt_y + C_yt_z \quad \Rightarrow \quad t_y = \frac{p_{24} - C_yt_z}{fK_y}$$

# Direct Linear Transformation Method

- Calculation of Distortion
  - A more complete model to the one used so far is the following, despising all errors except distortion.

$$x_f - D_x = \frac{p_{11}X_w + p_{12}Y_w + p_{13}Z_w + p_{14}}{p_{31}X_w + p_{32}Y_w + p_{33}z_w + p_{34}}$$

$$y_f - D_Y = \frac{p_{21}X_w + p_{22}Y_w + p_{23}Z_w + p_{24}}{p_{31}X_w + p_{32}Y_w + p_{33}Z_w + p_{34}}$$

$$x_f - x_{f\ \sin distorsion} = D_X = k_0 + k_1 x_f + k_2 y_f + k_3 x_f^2 + k_4 x_f y_f + k_5 y_f^2 + k_6 x_f^2 y_f + k_7 x_f y_f^2$$

undistorted

$$y_f - y_{f\ \sin distorsion} = D_Y = q_0 + q_1 y_f + q_2 x_f + q_3 y_f^2 + q_4 x_f y_f + q_5 x_f^2 + q_6 y_f^2 X_f + q_7 y_f x_f^2$$

undistorted

- It is important to calculate in separate steps the intrinsic and extrinsic parameters, and the distortion coefficients or other systematic errors coefficients to avoid instability.

- Distortion: Non-Linear Optimization Algorithm (minimizes reprojection error)

# Radial Alignment Constrain (RAC)

- Radial Alignment Constrain (RAC). Tsai Method
  - It employs a distortion model that only considers radial distortion.

$$x_u = x_d(k_1 r^2 + k_2 r^4 + \ldots.)$$
$$y_u = y_d(k_1 r^2 + k_2 r^4 + \ldots.)$$

With only Radial Distortion we get:

$$\frac{x_d}{y_d} = \frac{x_u}{y_u} = \frac{r_{11}x_w + r_{12}y_w + r_{13}z_w + t_x}{r_{21}x_w + r_{22}y_w + r_{23}z_w + t_y}$$

# Radial Alignment Constrain (RAC)

- Procedure:
  - It's supposed to be known $C_x$ , $C_y$ , $K_y$

  - Calculation of $|R|$ , $t_x$ , $t_y$
    - Calculate the undistorted central coordinates
    - Calculate the intermediate unknowns ($a_j$)
    - Calculate $|t_y|$
    - Calculate the sign of $t_y$
    - Calculate $S_x$
    - Calculate $|R|$ , $t_x$

  - Calculation of $f$ , $t_z$ , $k_1$
    - Initial approximation of $f$ , $t_z$
    - Iterative calculation of $f$ , $t_z$ , $k_1$

# Radial Alignment Constrain (RAC)

- Calculation of the undistorted central coordinates

From the points: $(x_f^i, y_f^i)$

We get $\begin{cases} \hat{x}_d^i = (x_f^i - C_x) = K_x x_d^i \\ \hat{y}_d^i = (y_f^i - C_y) = K_y y_d^i \end{cases}$

Building the quotient:

$$\frac{\hat{x}_d^i}{\hat{y}_d^i} = \frac{K_x x_d^i}{K_y y_d^i} = S_x \frac{x_u^i}{y_u^i}$$

with $S_x = \dfrac{K_x}{K_y}$

# Radial Alignment Constrain (RAC)

- Calculation of intermediate unknowns ($a_j$)

Replacing in the quotient

$$\frac{\hat{x}_d^i}{\hat{y}_d^i} = S_x \frac{x_u^i}{y_u^i} = S_x \frac{r_{11} x_w^i + r_{12} y_w^i + r_{13} z_w^i + t_x}{r_{21} x_w^i + r_{22} y_w^i + r_{23} z_w^i + t_y}$$

if $t_y \neq 0$

$$\hat{x}_d^i + \underbrace{t_y^{-1} r_{21}}_{a_4} x_w^i \hat{x}_d^i + \underbrace{t_y^{-1} r_{22}}_{a_5} y_w^i \hat{x}_d^i + \underbrace{t_y^{-1} r_{23}}_{a_6} z_w^i \hat{x}_d^i =$$

$$\underbrace{t_y^{-1} S_x r_{11}}_{a_1} x_w^i \hat{y}_d^i + \underbrace{t_y^{-1} S_x r_{12}}_{a_2} y_w^i \hat{y}_d^i + \underbrace{t_y^{-1} S_x r_{13}}_{a_3} z_w^i \hat{y}_d^i + \underbrace{t_y^{-1} S_x t_x}_{a_7} \hat{y}_d^i$$

# Radial Alignment Constrain (RAC)

- Calculation of intermediate unknowns ($a_j$)

A system with n equations and 7 unknowns is obtained:

$$\hat{x}_d^i = a_1 x_w^i \hat{y}_d^i + a_2 y_w^i \hat{y}_d^i + a_3 z_w^i \hat{y}_d^i - a_4 x_w^i \hat{x}_d^i - a_5 y_w^i \hat{x}_d^i - a_6 z_w^i \hat{x}_d^i + a_7 \hat{y}_d^i \hat{x}_d^i$$

- Calculation of $|t_y|$

$$t_y = \frac{1}{\sqrt{a_4^2 + a_5^2 + a_6^2}}$$

- Determining the sign of $t_y$

For any point (away from the center of the image) its projection is calculated without distortion, checking with the positive value and negative value

# Radial Alignment Constrain (RAC)

- Calculation of $S_x$

$$S_x = \sqrt{t_y^2(a_1^2 + a_2^2 + a_3^2)}$$

- Calculation of $|R|$ , $t_x$

$$r_{11} = a_1 \frac{t_y}{S_x} \qquad r_{12} = a_2 \frac{t_y}{S_x} \qquad r_{13} = a_3 \frac{t_y}{S_x}$$

$$r_{21} = a_4 t_y \qquad r_{22} = a_5 t_y \qquad r_{23} = a_6 t_y$$

$$r_{31} = \sqrt{1 - r_{11}^2 - r_{21}^2} \qquad r_{32} = \sqrt{1 - r_{12}^2 - r_{22}^2} \qquad r_{33} = \sqrt{1 - r_{13}^2 - r_{23}^2}$$

$$t_x = a_7 \frac{t_y}{S_x}$$

# Radial Alignment Constrain (RAC)

- Calculation of $f$ , $t_z$ , $k_1$
  - Initial approximation of $f$ , $t_z$
    - ***f* , *t*$_z$** are obtained without distortion by mean least squares

## Iterative calculation of $f$ , $k_1$ , $t_z$

- Nonlinear system optimization

$$\hat{x}_d^i = K_x k_1 \left[ \left( \hat{x}_d^i \right)^2 + \left( \hat{y}_d^i \right)^2 \right] + K_x f \frac{r_{11} x_w^i + r_{12} y_w^i + r_{13} z_w^i + t_x}{r_{31} x_w^i + r_{32} y_w^i + r_{33} z_w^i + t_z}$$

$$\hat{y}_d^i = K_y k_1 \left[ \left( \hat{x}_d^i \right)^2 + \left( \hat{y}_d^i \right)^2 \right] + K_y f \frac{r_{21} x_w^i + r_{22} y_w^i + r_{23} z_w^i + t_x}{r_{31} x_w^i + r_{32} y_w^i + r_{33} z_w^i + t_z}$$